Application to Population growth

$p(t)$ = population at any time t

$\frac{dp}{dt} \propto p \Rightarrow \frac{dp}{dt} = kp$

No. 1 (19.23) In a culture of yeast the rate of growth $y'(t)$ is proportional to the amount $y(t)$ present at time t. If $y(t)$ doubles in a day, how much population growth can be expected after 7 days

Sol^n Let $y(t)$ be population at any time t

$y = y_0$ at time $t = 0$ initial yeast population

Given $y = 2y_0$ at time $t = 1$ day

$y = ?$ after time $t = 7$ days.

$\frac{dy}{dt} \propto y$ (Rate of growth of population dy/dt is proportional to population $y(t)$)

$\frac{dy}{dt} = ky$ (k = constant of proportionality)

$\int \frac{dy}{y} = k\int dt \Rightarrow \ln y = kt + \ln c$

$y = ce^{kt}$ —— (1)

Given $y = y_0,\ t = 0 \Rightarrow y_0 = c$ —— (2)

Given $y = 2y_0,\ t = 1 \Rightarrow 2y_0 = ce^{k}$

$2y_0 = y_0 e^{k} \Rightarrow e^{k} = 2 \Rightarrow k = \ln 2$ —— (3)

$y = ce^{kt} = y_0 e^{kt} = y_0 (e^{k})^{t} = 2^{t} y_0$

$\boxed{y = 2^{t} y_0} \Rightarrow y = 2^{7} y_0 = 128\, y_0$ after $t = 7$ days

population = 128 times initial population y_0

Application to Newton's Law of Cooling

Rate of change of temperature of a body is directly proportional to temperature difference between the body & surroundings

Let T = temperature of the body at time t

T_s = temperature of the surrounding

Then $\frac{dT}{dt} \propto (T - T_s) \Rightarrow \frac{dT}{dt} = k(T - T_s)$

(13 / pg. 23) A thermometer reading $5°C$ is brought into a room whose temperature is $22°C$. One minute later the thermometer reading is $12°C$. How long does it take until the reading is practically $22°C$ (say $21.9°C$)

Sol^n^ Temp. $T = 5$ at time $t = 0$ $T_s = 22°C$

Temperature $T = 12$ after time $t = 1$ min.

Temperature $T = 21.9$ time $t = ?$ —— (1)

From Newton's law of cooling $\frac{dT}{dt} = k(T - T_s)$

$= k(T - 22)$

$$\int \frac{dT}{T-22} = k\int dt \Rightarrow \ln(T-22) = kt + \ln C$$

$$= \ln e^{kt} + \ln C = \ln Ce^{kt}$$

$$T - 22 = Ce^{kt}$$

$$T = 22 + Ce^{kt} \text{ —— (2)}$$

Given $T = 5$ at time $t = 0$ $\Rightarrow$ $C = -17$

$$T = 22 - 17e^{kt}$$

when ~~that~~ temperature $T = 12$ time $t = 1$

$$12 = 22 - 17e^{k} \Rightarrow e^{k} = \frac{10}{17}$$

$$T = 22 - 17e^{kt} = 22 - 17(e^{k})^{t} = 22 - 17\left(\frac{10}{17}\right)^{t}$$

when $T = 21.9 \Rightarrow 17\left(\frac{10}{17}\right)^{t} = 0.1$

$$t \ln\frac{10}{17} = \ln\frac{0.1}{17} \Rightarrow t = \frac{\ln(0.1/17)}{\ln(10/17)}$$

Application to RL Circuit

$$L\frac{dI}{dt} + RI = E(t) = EMF$$

Case - I $\quad L\frac{dI}{dt} + RI = E_0$ = Constant EMF

$$L\frac{dI}{dt} = E_0 - RI \Rightarrow \frac{dI}{E_0 - RI} = \frac{1}{L}dt$$

$$-\frac{1}{R}\ln(E_0 - RI) = \frac{1}{L}t + \ln c$$

$$\ln(E_0 - RI) = -\frac{Rt}{L} + \ln c_1 \Rightarrow E_0 - RI = c_1 e^{-\frac{Rt}{L}}$$

Case II $\quad L\frac{dI}{dt} + RI = E_0 \sin wt$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}\sin wt$$

Linear eqn with $P = \frac{R}{L}$ $\quad Q = \frac{E_0}{L}\sin wt$

$$F = e^{\int P\,dt} = e^{\frac{R}{L}\int dt} = e^{\frac{Rt}{L}}$$

Soln is $I = \frac{1}{F}\left[\int FQ\,dt + c\right]$

$$= e^{-Rt/L}\left[\int \frac{E_0}{L} e^{\frac{Rt}{L}} \sin wt\,dt + c\right]$$

$$= e^{-Rt/L}\left[\frac{E_0}{L}\frac{e^{Rt/L}}{\frac{R^2}{L^2} + w^2}\left(\frac{R}{L}\sin wt + w\cos wt\right) + c\right]$$

$$= \frac{E_0 L}{R^2 + w^2 L^2}\left(\frac{R}{L}\sin wt + w\cos wt\right) + c e^{-\frac{Rt}{L}}$$

exact

mixed