GRADIENT OF A SCALAR FIELD

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient f = grad f $= \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Physically grad f represents a vector normal to surface S in the space given by $f(x,y,z) = c$

(4) $f = x^2 + 9y^2$ Find grad f at $(-2,2)$

Sol^n grad $f = \hat{i}\frac{\partial}{\partial x}(x^2+9y^2) + \hat{j}\frac{\partial}{\partial y}(x^2+9y^2) + \hat{k}\frac{\partial}{\partial z}(x^2+9y^2)$

$= 2x\hat{i} + 18y\hat{j} = 2(-2)\hat{i} + 18(2)\hat{j}$ at $(-2,2)$

$= -4\hat{i} + 36\hat{j}$

(8) Find a vector normal to $f = \sin(x+z)$ at $(\frac{\pi}{8}, 1, \frac{\pi}{8})$

Sol^n Required normal vector is $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \cos(x+z)\hat{i} + 0 + \cos(x+z)\hat{k}$

$= \cos\frac{\pi}{4}\hat{i} + \cos(\frac{\pi}{4})\hat{k}$ at $(\frac{\pi}{8}, 1, \frac{\pi}{8}) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(20) Using gradient find the unit normal vector for $x^2 + y^2 + 2z^2 = 26$ at $P(2,2,3)$

Sol^n $f = x^2 + y^2 + 2z^2 - 26$

grad $f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$

$= 4\hat{i} + 4\hat{j} + 12\hat{k}$ at $(2,2,3)$

Required unit normal vector

$$= \frac{\text{grad } f}{|\text{grad } f|} = \frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{\sqrt{4^2 + 4^2 + 12^2}}$$

(21) $V = $ grad $f = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$, Find f ?

Sol^n $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 4y$ and $\frac{\partial f}{\partial z} = 8z$

$\partial f = 2x\,\partial x$, $\partial f = 4y\,\partial y$ and $\partial f = 8z\,\partial z$

$f = x^2 + c$, $f = y^2 + c$ and $f = z^2 + c$

$f = x^2 + y^2 + z^2 + c$

(22) $V = \text{grad } f = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\frac{\partial f}{\partial x} = yz$, $\frac{\partial f}{\partial y} = xz$ and $\frac{\partial f}{\partial z} = xy$

$\partial f = yz\,\partial x$, $\partial f = xz\,\partial y$ and $\partial f = xy\,\partial z$

$f = xyz + c$ $\quad f = xyz + c$ $\quad f = xyz + c$

(28) Prove that $\nabla(fg) = f\nabla g + g\nabla f$

<u>Proof</u> $\nabla(fg) = \hat{i}\frac{\partial}{\partial x}(fg) + \hat{j}\frac{\partial}{\partial y}(fg) + \hat{k}\frac{\partial}{\partial z}(fg)$

$= \hat{i}\left[f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right] + \hat{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \hat{k}\left[f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right]$

$= f\left[\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right] + g\left[\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right]$

$= f\nabla g + g\nabla f$

DIRECTIONAL DERIVATIVE

(31) Find the directional derivative of $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $P(3,0,4)$ along direction of vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

<u>Sol</u> $\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2x = -x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2}$

$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= -x(x^2+y^2+z^2)^{-3/2}\hat{i} - y(x^2+y^2+z^2)^{-3/2}\hat{j} - z(x^2+y^2+z^2)^{-3/2}\hat{k}$

$= -\frac{3}{(9+0+16)^{3/2}}\hat{i} - 0 - \frac{4}{(9+0+16)^{3/2}}\hat{k} = -\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}$

Required directional derivative at $P(3,0,4)$

$= (\text{grad } f)\cdot\frac{\vec{a}}{|a|} = \left(-\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}\right)\cdot\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{1^2+1^2+1^2}}$

$= \frac{-\frac{3}{125} + 0 - \frac{4}{125}}{\sqrt{3}} = -\frac{7}{125\sqrt{3}}$

(34) Find directional derivative of $f = x^2 + 3y^2 + 4z^2$ at $(1, 0, 1)$ along direction of vector $\vec{a} = -\hat{i} - \hat{j} + \hat{k}$

Sol: $\text{grad } f = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}\dfrac{\partial f}{\partial z}$

$= 2x\,\hat{i} + 6y\,\hat{j} + 8z\,\hat{k}$

$= 2\hat{i} + 0 + 8\hat{k}$ at $(1, 0, 1)$

Required directional derivative

$= \text{grad } f \cdot \dfrac{\vec{a}}{|a|}$

$= \dfrac{(2\hat{i} + 8\hat{k}) \cdot (-\hat{i} - \hat{j} + \hat{k})}{|-\hat{i} - \hat{j} + \hat{k}|}$

$= \dfrac{-2 + 0 + 8}{\sqrt{1+1+1}} = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$

DIVERGENCE OF A VECTOR FIELD

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Then $\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

(8) Find divergence of $\vec{V} = xyz(x\hat{i} + y\hat{j} + z\hat{k}$

Sol^n $\vec{V} = x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$

$\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 6xyz$

(6) Find divergence of $\vec{V} = (x^2+y^2+z^2)^{-3/2}(x\hat{i} + y\hat{j} + z\hat{k})$

Sol^n $\operatorname{div}\vec{V} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/2}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-3/2}\right]$
$+ \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-3/2}\right]$

$= (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}x(x^2+y^2+z^2)^{-5/2}2x$
$+ (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}y(x^2+y^2+z^2)^{-5/2}2y$
$+ (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}z(x^2+y^2+z^2)^{-5/2}2z$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$

(11) Check the flow with velocity $\vec{V} = y\hat{i}$ for incompressible

Sol^n $\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 0$

Hence the flow with velocity $y\hat{i}$ is incompressible

(13) prove that $\operatorname{div}(fv) = f\operatorname{div}v + v\cdot\nabla f$

RHS $= f\operatorname{div}v + v\cdot\nabla f$

$= f\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\cdot\partial f/\partial z\right)$

$= f\frac{\partial V_1}{\partial x} + f\frac{\partial V_2}{\partial y} + f\frac{\partial V_3}{\partial z} + V_1\frac{\partial f}{\partial x} + V_2\frac{\partial f}{\partial y} + V_3\frac{\partial f}{\partial z}$

$= f\frac{\partial V_1}{\partial x} + V_1\frac{\partial f}{\partial x} + f\frac{\partial V_2}{\partial y} + V_2\frac{\partial f}{\partial y} + f\frac{\partial V_3}{\partial z} + V_3\frac{\partial f}{\partial z}$

$= \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3) = \mathrm{div}(f v)$

13(c) Prove that $\mathrm{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

LHS $= \mathrm{div}(f \nabla g) = \mathrm{div}\left[i f \frac{\partial g}{\partial x} + j f \frac{\partial g}{\partial y} + k f \frac{\partial g}{\partial z} \right]$

$= \frac{\partial}{\partial x}\left(f \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f \frac{\partial g}{\partial z}\right)$

$= f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$

RHS $= f \nabla^2 g + \nabla f \cdot \nabla g$

$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}\right) \cdot \left(i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z}\right)$

$= f \frac{\partial^2 g}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} =$ LHS

(15) $f = 4x^2 + 9y^2 + z^2$ Find $\nabla^2 f$

Soln $\mathrm{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 8x\hat{i} + 18y\hat{j} + 2z\hat{k}$

$\nabla^2 f = \mathrm{div}(\mathrm{grad} f) = \mathrm{div}(8x\hat{i} + 18y\hat{j} + 2z\hat{k})$

$= \frac{\partial}{\partial x}(8x) + \frac{\partial}{\partial y}(18y) + \frac{\partial}{\partial z}(2z)$

$= 8 + 18 + 2 = 28$

CURL OF A VECTOR FIELD

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ then

$$\mathrm{curl}\,\vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

If $\mathrm{curl}\,\vec{V} = 0$ then the fluid motion with velocity $\vec{V}$ is irrotational.

③ Find curl $\vec{v}$ where $\vec{v} = \frac{x^2+y^2+z^2}{2}(\hat{i}+\hat{j}+\hat{k})$

Sol^n

$$\text{curl}\,\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x^2+y^2+z^2}{2} & \frac{x^2+y^2+z^2}{2} & \frac{x^2+y^2+z^2}{2} \end{vmatrix}$$

$$= \hat{i}\,\frac{1}{2}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x^2+y^2+z^2)\right]$$

$$- \hat{j}\,\frac{1}{2}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x^2+y^2+z^2)\right] + \hat{k}.\frac{1}{2}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial y}(x^2+y^2+z^2)\right]$$

$$= (y-z)\,\hat{i} - (x-z)\,\hat{j} + (x-y)\,\hat{k}$$

⑪ Check $\vec{v} = y\hat{i} - x\hat{j}$ for irrotational ? incompressible?

Sol^n $\text{div}\,\vec{v} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0 - 0 = 0$

hence incompressible.

$$\text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 2\hat{k} = -2\hat{k} \neq 0$$

Hence $\vec{v}$ is not irrotational.

14(b) Prove that $\text{div}(\text{curl}\,\vec{v}) = 0$

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\text{curl}\,\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$\text{div}(\text{curl}\,\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x\partial y} - \frac{\partial^2 v_2}{\partial x\partial z} - \frac{\partial^2 v_3}{\partial x\partial y} + \frac{\partial^2 v_1}{\partial y\partial z} + \frac{\partial^2 v_2}{\partial x\partial z} - \frac{\partial^2 v_1}{\partial y\partial z}$$

$$= 0$$

14(d) Prove that curl (grad f) $= \vec{0}$

Sol^n By definition grad $f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$\text{curl}(\text{grad } f) = \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{k}$$

$$= 0\,\hat{i} - 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

(17) $u = y\hat{i} + z\hat{j} + x\hat{k}$, $v = yz\hat{i} + zx\hat{j} + xy\hat{k}$ Find $\vec{u} \times \text{curl}\,\vec{v}$

Sol^n

$$\text{curl}\,\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= (x - x)\hat{i} - (y - y)\hat{j} + (z - z)\hat{k} = \vec{0}$$

Hence $\vec{u} \times \text{curl}\,\vec{v} = \vec{0}$

(19) $u = y\hat{i} + z\hat{j} + x\hat{k}$, $v = yz\hat{i} + zx\hat{j} + xy\hat{k}$ Find $\vec{v} \cdot \text{curl}\,\vec{u}$

Sol^n

$$\text{curl}\,\vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right]$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

$$\vec{v} \cdot \text{curl}\,\vec{u} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot (-\hat{i} - \hat{j} - \hat{k})$$

$$= -yz - zx - xy$$

9.1 LINE INTEGRAL

MODULE-5

Parametric Representation

(A) • Line Segment joining points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

$r(t) = a + bt$ where $\vec{a} = \vec{OA}$ = position vector of A and $\vec{b} = \vec{AB}$

$0 \le t \le 1$

(B) **Circle** $(x-h)^2 + (y-k)^2 = a^2$

Put $x - h = a\cos t$, $y - k = a\sin t$, $z = 0$

$r(t) = x\hat{i} + y\hat{j} + z\hat{k} = (h + a\cos t)\hat{i} + (k + a\sin t)\hat{j}$

$0 \le t \le 2\pi$

(C) **Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Put $x = a\cos t$, $y = b\sin t$)

$r(t) = x\hat{i} + y\hat{j} + z\hat{k} = a\cos t\,\hat{i} + b\sin t\,\hat{j}$, $0 \le t \le 2\pi$

(D) **Parabola** $y^2 = 4ax$ (Put $x = at^2$, $y = 2at$, $z = 0$)

$r(t) = x\hat{i} + y\hat{j} + z\hat{k} = at^2\,\hat{i} + 2at\,\hat{j}$

(E) **Any Curve** $y = f(x)$ $z = g(x)$

Let $x = t$ then $y = f(t)$ $z = g(t)$

$r(t) = x\hat{i} + y\hat{j} + z\hat{k} = t\hat{i} + f(t)\hat{j} + g(t)\hat{k}$

No. 1 / Pg. 470 Find $\int_C F \cdot dr$, $F = [y^2, -x^2]$, C: straight line segment from $(0,0)$ to $(1,4)$

Sol $A(0,0)$, $B(1,4)$, $\vec{b} = \vec{AB} = \hat{i} + 4\hat{j}$, $\vec{a} = \vec{OA} = \vec{0}$

Parametric form is $r(t) = a + bt = (\hat{i} + 4\hat{j})t$, $0 \le t \le 1$

$dr = (\hat{i} + 4\hat{j})\,dt$

$r(t) = t\hat{i} + 4t\hat{j} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow x = t$, $y = 4t$

$\vec{F} = y^2\hat{i} - x^2\hat{j} = 16t^2\,\hat{i} - t^2\hat{j}$

$\int_C F \cdot dr = \int_0^1 (16t^2\,\hat{i} - t^2\hat{j}) \cdot (\hat{i} + 4\hat{j})\,dt = \int_0^1 (16t^2 - 4t^2)\,dt$

$= \left[12\frac{t^3}{3}\right]_0^1 = 4$

No. 2 / Pg. 470 Evaluate $\int_C F \cdot dr$, $F = [y^2, -x^2]$, C: Parabola $y = 4x^2$ $(0,0)$ to $(1,4)$

Sol Given parabola is $y = 4x^2$

Let $x = t$ then $y = 4t^2$ $z = 0$

$r(t) = x\hat{i} + y\hat{j} = t\hat{i} + 4t^2\hat{j} \Rightarrow dr = \hat{i} + 8t\hat{j}$

$\vec{F} = y^2\hat{i} - x^2\hat{j} = 16t^4\,\hat{i} - t^2\hat{j}$ $\quad 0 \le t \le 1$

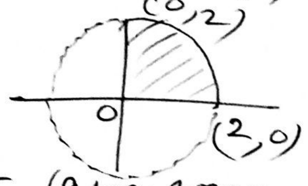
$$\int_C F\cdot dr = \int_0^1 \left(16t^4\,\hat{i} - t^2\,\hat{j}\right)\cdot\left(\hat{i} + 8t\,\hat{j}\right)dt$$

$$= \int_0^1 \left(16t^4 - 8t^3\right)dt = \left[16\frac{t^5}{5} - 8\frac{t^4}{4}\right]_0^1 = \frac{6}{5}$$

No. 3 Pg. 470 Evaluate $\int_C F\cdot dr$, $F = [xy,\ x^2y^2]$

C: quarter circle from $(2,0)$ to $(0,2)$ with centre $(0,0)$

Sol^n C: Circle $x^2+y^2 = 2^2$ (1st quadrant)



Put $x = 2\cos t$ $y = 2\sin t$ $z = 0$

$r(t) = x\hat{i} + y\hat{j} + z\hat{k} = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$

$dr = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$ $0 \le t \le \frac{\pi}{2}$ (quarter circle)

$\vec{F} = xy\,\hat{i} + x^2y^2\,\hat{j} = 4\sin t\cos t\,\hat{i} + 16\sin^2 t\cos^2 t\,\hat{j}$

$$\int_C F\cdot dr = \int_0^{\pi/2}\left(-8\sin^2 t\cos t + 32\sin^2 t\cos^3 t\right)dt$$

$$= \int_0^{\pi/2}\left[-8\sin^2 t + 32\sin^2 t\left(1-\sin^2 t\right)\right]\cos t\,dt$$

put $\sin t = u$
$\cos t\,dt = du$
As $t \to 0$ $u \to 0$
$t \to \frac{\pi}{2}$ $u \to 1$

$$= \int_0^1\left[-8u^2 + 32u^2\left(1-u^2\right)\right]du$$

$$= \left[-8\frac{u^3}{3} + 32\frac{u^3}{3} - 32\frac{u^5}{5}\right]_0^1 = -\frac{8}{3} + \frac{32}{3} - \frac{32}{5} = 8/5$$

No. 13 $\int_C f(r)\,ds$, $f = x^2+y^2$, C: $y = 3x$, from $(0,0)$ to $(2,6)$

Sol^n $A(0,0)$ $B(2,6)$, $\vec{a} = \vec{OA} = \vec{0}$, $\vec{b} = \vec{AB} = 2\hat{i} + 6\hat{j}$

$r(t) = a + bt = (2\hat{i} + 6\hat{j})t \Rightarrow \frac{dr}{dt} = 2\hat{i} + 6\hat{j}$

$$\frac{ds}{dt} = \sqrt{\frac{dr}{dt}\cdot\frac{dr}{dt}} = \sqrt{(2\hat{i}+6\hat{j})\cdot(2\hat{i}+6\hat{j})} = \sqrt{40}$$ $0 \le t \le 1$

$ds = \sqrt{10}\,dt$

$r(t) = 2t\,\hat{i} + 6t\,\hat{j} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow x = 2t,\ y = 6t,\ z = 0$

$f = x^2+y^2 = 4t^2 + 36t^2 = 40t^2$

$$\int_C f(r)\,ds = \int_0^1 40t^2\sqrt{10}\,dt = 40\sqrt{10}\left[\frac{t^3}{3}\right]_0^1 = \frac{40\sqrt{10}}{3}$$

9.2 Line Integral Independent of Path

$r = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \Rightarrow \int_C F\cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$

$F_1 = \frac{\partial f}{\partial x} \quad F_2 = \frac{\partial f}{\partial y} \quad F_3 = \frac{\partial f}{\partial z}$

$F = \text{grad } f = F_1 dx + F_2 dy + F_3 dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

A line integral is path independent (or exact) if curl $\vec{F} = 0$

No. 6 Show that for $I = \int_{(0,0,0)}^{(4,1,2)} (3y\,dx + 3x\,dy + 2z\,dz)$ the form under integral sign is exact & evaluate.

Sol^n $F_1 = 3y$ = coeff. of dx $\quad F_2 = 3x$ = coeff of dy
$F_3 = 2z$ = Coeff of dz

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 3x & 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(3x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial z}(3y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(3y)\right]$$

$= 0\hat{i} - 0\hat{j} + (3-3)\hat{k} = \vec{0}$ Hence ~~integrand~~ integrand is exact & the line integral is path independent.

$F_1 = \frac{\partial f}{\partial x} = 3y$ —— (1) $\quad F_2 = \frac{\partial f}{\partial y} = 3x$ —— (2) and $F_3 = \frac{\partial f}{\partial z} = 2z$ —— (3)

From eq^n(1) $\int \partial f = 3y\int \partial x \Rightarrow f = 3xy + c(y,z)$ —— (4)

Differentiating partially w.r.to y & z respectively we get $\frac{\partial f}{\partial y} = 3x + \frac{\partial c}{\partial y}$ —— (5) and $\frac{\partial f}{\partial z} = \frac{\partial c}{\partial z}$ —— (6)

From eq^n (2) & eq^n (5) $3x = 3x + \frac{\partial c}{\partial y} \Rightarrow \frac{\partial c}{\partial y} = 0$
c is independent of y

From eq^n (3) & eq^n(6) $\frac{\partial c}{\partial z} = 2z \Rightarrow \int \partial c = 2\int z\,\partial z$
$c = z^2$

Hence $f = 3xy + z^2$

$I = [f(x,y,z)]_{(0,0,0)}^{(4,1,2)} = f(4,1,2) - f(0,0,0) = 16 - 0$
$= 16$

9.3 DOUBLE INTEGRAL

$$\int_{x=a}^{b}\int_{y=g(x)}^{h(x)} f(x,y)\,dy\,dx \quad \text{OR} \quad \int_{y=c}^{d}\int_{x=p(y)}^{q(y)} f(x,y)\,dx\,dy$$

volume $V=\iint_R f\,dx\,dy$ $\qquad M = \iint_R f(x,y)\,dx\,dy$

C.G. $\bar{x} = \frac{1}{M}\iint x f(x,y)\,dx\,dy$, $\bar{y} = \frac{1}{M}\iint y f(x,y)\,dx\,dy$

M.I. $I_x = \iint_R y^2 f(x,y)\,dx\,dy$, $I_y = \iint_R x^2 f(x,y)\,dx\,dy$

Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

No.5 $\int_0^{\pi/4}\int_0^{y} \frac{\sin y}{y}\,dx\,dy = \int_0^{\pi/4} \frac{\sin y}{y}\,[x]_0^y\,dy$

$= \int_0^{\pi/4} \frac{\sin y}{y}\cdot y\,dy = \int_0^{\pi/4} \sin y\,dy = -\cos y\Big]_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}}$

No.9 $\int_1^5 \int_0^{x^2} (1+2x)e^{x+y}\,dy\,dx = \int_1^5\int_0^{x^2} e^y\cdot e^x(1+2x)\,dy\,dx$

$= \int_1^5 \left[\int_0^{x^2} e^y\,dy\right](1+2x)e^x\,dx = \int_1^5 (e^{x^2}-1)(1+2x)e^x\,dx$

$= \int_1^5 e^{x^2+x}(1+2x)\,dx - \int (1+2x)e^x\,dx$

$= \int e^u\,du - \left[(1+2x)e^x - \int 2e^x\,dx\right]$

put $x^2+x = u$
$(1+2x)\,dx = du$
in first integral

$= e^u - \left[e^x + 2xe^x - 2e^x\right] = e^{x^2+x} \cancel{- (2x+1)e^x} - (2x-1)e^x$

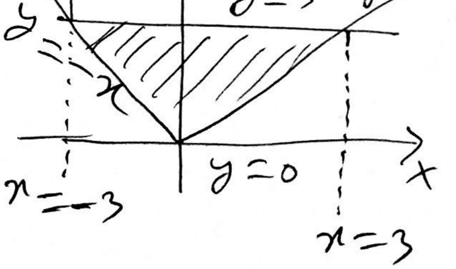
Change of order

No.4 By change of order solve $\int_0^3 \int_{-y}^{y} (x^2+y^2)\,dx\,dy$

Sol Here the boundary regions are

$x=-y$, $x=y$, $y=0$ and $y=3$

The region is shown in figure.



Hence $y=-x$, $y=x$, $x=-3$, $x=3$
(by solving we get)

when $-3 < x \leq 0$ the region is bounded by lines $y = -x$ and $y = 3$

Also when $0 < x \leq 3$ the region is bounded by lines $y = x$ and $y = 3$

Hence after change of order given integral becomes

$$\int_{x=-3}^{0} \int_{y=-x}^{3} (x^2+y^2)\,dy\,dx + \int_{x=0}^{3} \int_{y=x}^{3} (x^2+y^2)\,dy\,dx$$

$$= \int_{-3}^{0} \left[x^2 y + \frac{y^3}{3}\right]_{y=-x}^{3} dx + \int_{0}^{3} \left[x^2 y + \frac{y^3}{3}\right]_{y=x}^{3} dx$$

$$= \int_{-3}^{0} \left(3x^2 + 9 + x^3 + \frac{x^3}{3}\right) dx + \int_{0}^{3} \left(3x^2 + 9 - x^3 - \frac{x^3}{3}\right) dx$$

$$= \left[x^3 + 9x + \frac{x^4}{4} + \frac{x^4}{12}\right]_{-3}^{0} + \left[x^3 + 9x - \frac{x^4}{4} - \frac{x^4}{12}\right]_{0}^{3} = 54$$

(12) Find volume of 1^{st} octant bounded by coordinate planes and surfaces $y = 1 - x^2$, $z = 1 - x^2$

Sol^n $x = 0$, $y = 0$, $y = 1 - x^2$, Given $z = f(x,y) = 1 - x^2$

$1 - x^2 = 0 \Rightarrow x = \pm 1$ We take $x = 1$ (1^{st} quadrant)

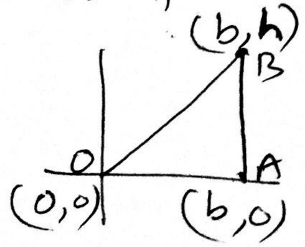
Hence $0 < x \leq 1$, $0 < y \leq 1 - x^2$

$$\text{Volume} = \iint_R z\,dx\,dy = \int_0^1 \int_0^{1-x^2} (1-x^2)\,dx\,dy$$

$$= \int_0^1 (1-x^2)\left[y\right]_0^{1-x^2} dx = \int_0^1 (1-x^2)(1-x^2)\,dx$$

$$= \int_0^1 (1 + x^4 - 2x^2)\,dx = x + \frac{x^5}{5} - \frac{2}{3}x^3 \Big]_0^1 = 1 + \frac{1}{5} - \frac{2}{3} = \frac{8}{15}$$

(14) Find coordinates $\bar{x}$, $\bar{y}$ of C.G. of a mass of density $f(x,y) = 1$ in region R: Δ with vertices $(0,0)$, $(b,0)$, (b,h)



Sol^n $f(x,y) = 1$ $O(0,0)$ $B(b,h)$

Eq^n of line OB is $y - 0 = \dfrac{h-0}{b-0}(x - 0)$

$y = \dfrac{hx}{b}$

$0 < x \le b \;,\; 0 < y \le \frac{hx}{b}$

$M = \iint_R f(x,y)\,dx\,dy = \int_0^b \int_0^{hx/b} dx\,dy = \int_0 [y]_0^{\frac{hx}{b}} dx$

$= \int_0^b \frac{h}{b} x\,dx = \frac{h}{b}\frac{x^2}{2}\Big]_0^b = \frac{h}{2b}\cdot b^2 = \frac{hb}{2}$

$\bar{x} = \frac{1}{M}\int_0^b \int_0^{hx/b} x f(x,y)\,dx\,dy = \frac{2}{hb}\int_0^b \int_0^{hx/b} x\,dx\,dy$

$= \frac{2}{hb}\int_0^b x\,[y]_0^{hx/b}\,dx = \frac{2}{hb}\int_0^b x\cdot\frac{hx}{b}\,dx$

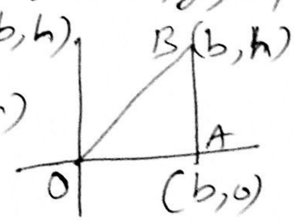
$= \frac{2}{hb}\frac{h}{b}\frac{x^3}{3}\Big]_0^b = \frac{2}{3b^2}b^3 = \frac{2b}{3}$

$\bar{y} = \frac{1}{M}\int_0^b \int_0^{hx/b} y f(x,y)\,dx\,dy = \frac{2}{hb}\int_0^b \int_0^{hx/b} y\,dx\,dy$

$= \frac{2}{hb}\int_0^b \left[\frac{y^2}{2}\right]_0^{hx/b} dx = \frac{2}{hb}\int_0^b \frac{h^2}{2b^2} x^2\,dx$

$= \frac{2}{hb}\frac{h^2}{2b^2}\frac{x^3}{3}\Big]_0^b = \frac{h}{3b^3}b^3 = \frac{h}{3}$

(17) Find M.I. I_x, I_y of a mass of density $f(x,y)=1$ in region R: Δ with vertices $(0,0)$, $(b,0)$, (b,h),



Sol^n Eq^n of OB is $y = hx/b$, $f(x,y)=1$ (given)

$0 < x \le b \quad 0 < y \le hx/b$

$I_x = \iint_R y^2 f(x,y)\,dx\,dy = \int_{x=0}^{b}\int_{y=0}^{hx/b} y^2\,dx\,dy$

$= \int_0^b \left[\frac{y^3}{3}\right]_0^{hx/b} dx = \int_0^b \frac{h^3}{3b^3} x^3\,dx = \left[\frac{h^3}{3b^3}\frac{x^4}{4}\right]_0^b = \frac{bh^3}{12}$

$I_y = \iint_R x^2 f(x,y)\,dx\,dy = \int_0^b \int_0^{hx/b} x^2\,dx\,dy$

$= \int_0^b x^2\,[y]_0^{hx/b}\,dx = \int_0^b x^2\frac{hx}{b}\,dx = \frac{h}{b}\frac{x^4}{4}\Big]_0^b = \frac{hb^3}{4}$

9.4 GREEN'S THEOREM Let R be a closed bounded region in XY-plane whose boundary C consists of infinitely many smooth curves. Let $F_1(x,y)$, $F_2(x,y)$ be functions that are continuous & have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ & $\frac{\partial F_2}{\partial x}$ every where in same domain containing R then

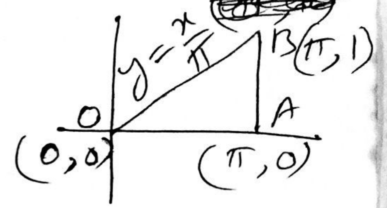
$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

Formula Area $A = \frac{1}{2}\oint (x\,dy - y\,dx) = \frac{1}{2}\oint r^2\,d\theta$

No. 6 Using Greens Th^m evaluate $\oint_C \vec{F}\cdot d\vec{r}$

$\vec{F} = [\sin y, \cos x]$ R: Δ with vertices $(0,0)$, $(\pi,0)$ & $(\pi,1)$

Sol^n $O(0,0)$ $B(\pi,1)$



Eq^n of line OB is $y = x/\pi$

$0 < x \leq \pi$, $0 < y \leq \frac{x}{\pi}$

$F = \sin y\,\hat{i} + \cos x\,\hat{j} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$F_1 = \sin y$ = coeff of $\hat{i}$ $F_2 = \cos x$ = coeff of $\hat{j}$

$F_3 = 0$ = coeff of $\hat{k}$ $\frac{\partial F_1}{\partial y} = \cos y$ $\frac{\partial F_2}{\partial x} = -\sin x$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (F_1\hat{i} + F_2\hat{j} + F_3 k)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_C F_1\,dx + F_2\,dy + F_3\,dz = \int_C F_1\,dx + F_2\,dy$$

(Since $F_3 = 0$)

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \text{(Using Green's Th}^m\text{)}$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{x/\pi} (-\sin x - \cos y)\,dx\,dy$$

$$= \int_0^{\pi}\left[-\sin x\int_0^{x/\pi} dy - \int_0^{x/\pi}\cos y\,dy\right] dx$$

$$= \int_0^{\pi}\left[-y\sin x - \sin y\right]_0^{x/\pi} dx$$

$$= \int_0^{\pi} \left[-\frac{x}{\pi} \sin x - \sin \frac{x}{\pi} \right] dx$$

$$= -\frac{1}{\pi} \left[-x\cos x + \sin x \right]_0^{\pi} + \pi \cos \frac{x}{\pi} \Big]_0^{\pi}$$

$$= -\frac{1}{\pi} \left[\pi - 0 + 0 - 0 \right] + \pi \cos 1 - \pi = -1 - \pi + \pi \cos 1$$

(11) Find area of Cardioid $r = a(1-\cos\theta)$ in the first quadrant.

Sol Area $= \frac{1}{2} \oint r^2 d\theta = \frac{1}{2} \int_0^{\pi/2} a^2 (1-\cos\theta)^2 d\theta$

$$= \frac{a^2}{2} \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left[1 - 2\cos\theta + \frac{1+\cos 2\theta}{2} \right] d\theta$$

$$= \frac{a^2}{2} \left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{2} - 2 + \frac{\pi}{4} + 0 - 0 - 0 - 0 - 0 \right] = \left(\frac{3\pi}{8} - 1 \right) a^2$$

9.5 SURFACES FOR SURFACE INTEGRAL

$$r(u,v) = x\hat{i} + y\hat{j} + z\hat{k}$$

(1) xy plane $x = u,\ y = v,\ z = 0,\ \vec{r} = u\hat{i} + v\hat{j}$

(2) XY-plane (polar) $x = u\cos v,\ y = u\sin v,\ z = 0$

$\vec{r} = u\cos v\,\hat{i} + u\sin v\,\hat{j}$

(3) Cone $z = c\sqrt{x^2+y^2}$ (put $x = u\cos v,\ y = u\sin v$ then $z = cu$)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = u\cos v\,\hat{i} + u\sin v\,\hat{j} + cu\hat{k}$

(4) paraboloid $z = x^2 + y^2$ (put $x = u\cos v,\ y = u\sin v$ then $z = u^2$)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\hat{k}$

(5) Elliptic parabolloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = au\cos v\,\hat{i} + bu\sin v\,\hat{j} + u^2\hat{k}$

(6) Sphere $x^2 + y^2 + z^2 = a^2$ put $x = a\cos v\cos u$.

$y = a\cos v\sin u$ $z = a\sin v$