Complex valued Function MODULE - 3

$f : \mathbb{C} \to \mathbb{C}$ where C: set of complex numbers.

A function f defined on C is a rule that assign to every z in C a complex number w called the value of f at z.

Limit $\lim_{z \to z_0} f(z) = l$ if $|f(z) - l| < \epsilon$ for $|z - z_0| < \delta$

Continuity $\lim_{z \to z_0} f(z) = f(z_0)$ for f(z) continuous at point z_0

Differentiation $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

All the rules like product rule, chain rule etc are same as in real calculus.

(20) Find the value of derivative of $f(z) = (3z^2 + iz)^2$ at $1+i$

Sol:

$f'(z) = 2(3z^2 + iz)(6z + i)$

$f'(1+i) = 2[3(1+i)^2 + i(1+i)](6 + 6i + i)$

$= 2[-12 + i - 1][6 + 7i)$

Analytic Function A function $f(z)$ is called analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

Also $f(z)$ is analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighbourhood of z_0.

Cauchy – Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{where } f(z) = u(x,y) + i\,v(x,y)$$

$u = \operatorname{Re} z$, $v = \operatorname{Im} z$

Cauchy – Riemann eq^n s in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

No. 4 Check $f(z) = i/z^5$ for analytic?

Sol^n Let $z = x+iy$, $f(z) = u + iv$

$z = x+iy = r(\cos\theta + i\sin\theta)$

$$f(z) = \frac{i}{z^5} = i z^{-5} = r^{-5}\left[\cos\theta + i\sin\theta\right]^{-5}$$

$$= r^{-5}\left[\cos 5\theta - i\sin 5\theta\right] = u + iv$$

$u = r^{-5}\cos 5\theta$ = real part of $f(z)$

$v = -r^{-5}\sin 5\theta$ = Imaginary part of $f(z)$

Now $$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(-5r^{-5}\cos 5\theta\right) = -5r^{-6}\cos 5\theta = \partial u/\partial r$$

Also $$-\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}\left(-5r^{-5}\sin 5\theta\right) = 5r^{-6}\sin 5\theta = \frac{\partial v}{\partial r}$$

Cauchy – Riemann eq^ns are satisfied.

Hence $f(z) = i/z^5$ is analytic.

No. 21 Determine whether $u = -e^{-x}\sin y$ is harmonic. If yes, find the corresponding analytic function $f(z) = u(x,y) + i\,v(x,y)$

Sol^n $u_x = \frac{\partial u}{\partial x} = e^{-x}\sin y$, $u_{xx} = \frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y$

$u_y = \frac{\partial u}{\partial y} = -e^{-x}\cos y$, $u_{yy} = \frac{\partial^2 u}{\partial y^2} = e^{-x}\sin y$

Adding we get $u_{xx} + u_{yy} = 0$

Hence u is harmonic. (satisfies Laplace eq^n)

To find corresponding analytic function

$f(z) = u + iv$, Cauchy-Riemann eq^n's are satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = e^{-x}\sin y \quad \text{(1)} \quad \text{and} \quad \frac{\partial v}{\partial x} = e^{-x}\cos y \quad \text{(2)}$$

From eq^n (1)

$\Rightarrow \partial v = e^{-x}\sin y\, \partial y \Rightarrow \int \partial v = e^{-x}\int \sin y\, dy$

$v = -e^{-x}\cos y + c(x)$

$$\frac{\partial v}{\partial x} = e^{-x}\cos y + \frac{\partial c}{\partial x} \quad \text{(3)}$$

From (2) & (3) $\frac{\partial c}{\partial x} = 0 \Rightarrow c(x) = c = \text{const}$

$v = -e^{-x}\cos y + c$

$f(z) = u + iv = -e^{-x}\sin y + i(-e^{-x}\cos y + c)$

$= -e^{-x}(\sin y + i\cos y) + ic$

$= -ie^{-x}(\cos y - i\sin y) + ic = -ie^{-x}e^{-iy} + ic$

$= -ie^{-(x+iy)} + ic = -ie^{-z} + ic$

No. 25 Find a, b such that $u = ax^3 + by^3$ harmonic

Sol^n ~~For harmoni~~ For u to be harmonic $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow 6ax + 6by = 0$ — (1)

Hence $a = 0$, $b = 0$ is required to satisfy eq^n (1) so that u ~~is~~ will be harmonic.

Conjugate harmonic: If two harmonic functions u and v satisfy Cauchy-Riemann equations in a domain D they are the real & imaginary parts of an analytic function f in D. Then v is said to be a conjugate harmonic function of u in D

EXPONENTIAL FUNCTION $z = x + iy$

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$$

e^z is analytic for all z.

Euler Formula $e^{iy} = \cos y + i \sin y$

$$e^{-iy} = \cos y - i \sin y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

(exponential polar form)

$$e^{2n\pi i} = 1, \quad e^{\pi i} = -1, \quad e^{-\pi i} = -1, \quad e^{\frac{\pi}{2} i} = i, \quad e^{-\frac{\pi}{2} i} = -i$$

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

No.1 Find e^z, $z = 2 + 3\pi i$

Sol $e^z = e^{2+3\pi i} = e^2[\cos 3\pi + i \sin 3\pi] = -e^2$

No.6 Express $1+i$ in exponential polar form

Sol^n^ $1 + i = x + iy \Rightarrow x = 1 \quad y = 1$

$r = \sqrt{x^2 + y^2} = \sqrt{2} \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1} 1 = \frac{\pi}{4}$

Exponential polar form $z = re^{i\theta} = \sqrt{2}\, e^{i\frac{\pi}{4}}$

No.10 Represent $\sqrt{i}$ in exponential polar form

Sol Let $z = \sqrt{i} \Rightarrow z^2 = i = x + iy$

$x = 0, y = 1 \Rightarrow r = \sqrt{x^2 + y^2} = 1, \theta = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$

$$z^2 = re^{i\theta} = e^{i\pi/2} \Rightarrow z = e^{i\pi/4}$$

NO. 12 Find all the solutions of $e^{2z} = 2$

Soln Let $z = x + iy$

$e^{2z} = e^{2x+2iy} = e^{2x} \times e^{2iy} = e^{2x}(\cos 2y + i \sin 2y) = 2$

$\Rightarrow e^{2x}\cos 2y = 2$ — (1) and $e^{2x}\sin 2y = 0$ — (2)

(Equating real and imaginary parts)

Squaring & adding $e^{2x}[\cos^2 2y + \sin^2 2y] = 4$

$e^{2x} = 4 \Rightarrow 2x = \ln 4 = 2\ln 2 \Rightarrow x = \ln 2$

$\frac{Eq^n (2)}{Eq^n (1)} \Rightarrow \frac{\sin 2y}{\cos 2y} = \frac{0}{2} \Rightarrow \tan 2y = 0$

$2y = \cancel{\pm 2n\pi} \pm 2n\pi$

$z = x + iy = \ln 2 \pm n\pi i$ where $n = 1, 2, 3, \ldots$

NO-13 Find all solutions of $e^z = -3$

Soln Let $z = x + iy$

$e^z = e^{x+iy} = e^x(\cos y + i \sin y) = -3$

Equating real and imaginary parts we get

$e^x \cos y = -3$ — (3) and $e^x \sin y = 0$ — (4)

Squaring & adding eq^n(3) & eq^n(4) we get

$(e^x)^2(\cos^2 y + \sin^2 y) = 9 \Rightarrow (e^x)^2 = 9$

$e^x = 3 \Rightarrow x = \ln 3$

$\frac{e^x \sin y}{e^x \cos y} = \frac{0}{-3} \Rightarrow \tan y = \frac{0}{-3}$

$y = \pm(2n+1)\pi$

$z = x + iy = \ln 3 \pm (2n+1)\pi i$

TRIGONOMETRIC & HYPERBOLIC FUNCTIONS

$e^{iz} = \cos z + i \sin z$, $e^{-iz} = \cos z - i \sin z$

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

$\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$\cancel{\cosh z = \frac{1}{2}(e^{iz} + e^{-iz})}$, $\cancel{\sinh z = \frac{1}{2}(e^{i}}$

$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$, $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$

$(\cosh z)' = \sinh z$, $(\sinh z)' = \cosh z$

$\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$

$\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$

Next prove that $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$\sinh z = \sinh x \cos y + i \cosh x \sin y$

Proof $RHS = \frac{e^{x} + e^{-x}}{2} \times \frac{e^{iy} + e^{-iy}}{2} + i \frac{e^{x} - e^{-x}}{2} \times \frac{e^{iy} - e^{-iy}}{2i}$

$= \frac{1}{4}\left[e^{x+iy} + e^{x-iy} + e^{-x+iy} + e^{-x-iy} + e^{x+iy} - e^{x-iy} - e^{-x+iy} + e^{-x-iy}\right]$

$= \frac{1}{4}\left[2e^{x+iy} + 2e^{-x-iy}\right] = \frac{1}{4}(2e^{z} + 2e^{-z})$

$= \frac{e^{z} + e^{-z}}{2} = \cosh z = LHS$ where $z = x + iy$

Similarly $RHS = \frac{e^{x} - e^{-x}}{2} \times \frac{e^{iy} + e^{-iy}}{2} + i \frac{e^{x} + e^{-x}}{2} \times \frac{e^{iy} - e^{-iy}}{2i}$

$= \frac{1}{4}\left[e^{x+iy} - e^{-x+iy} + e^{x-iy} - e^{-x-iy} + e^{x+iy} + e^{-x+iy} - e^{x-iy} - e^{-x-iy}\right]$

$= \frac{1}{4}\left[2e^{x+iy} - 2e^{-x-iy}\right) = \frac{1}{4}\left[2e^{z} - 2e^{-z}\right] = \frac{e^{z} - e^{-z}}{2}$

$= \sinh z$

No. 6 Find the value of $\cos\left(\frac{\pi}{2} - \pi i\right)$ in terms of $u + iv$

Sol^n $\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

$\cos\left(\frac{\pi}{2} - \pi i\right) = \cos\frac{\pi}{2}\cosh(-\pi) - i \sin\frac{\pi}{2} \sinh(-\pi)$

$= 0 - i \sinh(-\pi) = i \sinh \pi$ (put $x = \frac{\pi}{2}, y = -\pi$)

No. 12 Find all the solutions of $\sin z = 1000$

Sol^n Let $z = x + iy$

$\sin z = \sin x \cosh y + i \cos x \sinh y = 1000$

$\Rightarrow \sin x \cosh y = 1000$ —— (1) and $\cos x \sinh y = 0$ —— (2)

(Equating real & Imag. parts)

From (2) $\cos x = 0$ Or $\sinh y = 0$

$x = \pm(2n+1)\frac{\pi}{2}$ $\Rightarrow \frac{e^y - e^{-y}}{2} = 0$

$= \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\ldots$ $\Rightarrow y = 0$

Putting $x = \frac{\pi}{2}$ in eq^n (1) $\cosh y = 1000$

$y = \cosh^{-1}(1000)$

Putting $y = 0$ in eq^n (1) $\sin x = 1000$ (impossible)

(as $-1 \le \sin x \le 1$). Hence we discard $y = 0$

Solution is $z = x + iy = \pm(2n+1)\frac{\pi}{2} + i\cosh^{-1}(1000)$

No. 9 Find all solutions of $\cos z = 3i$

Sol^n $\cos z = \cos x \cosh y - i \sin x \sinh y = 3i$

Equating real & imaginary parts

$\cos x \cosh y = 0$ —— (3) and $\sin x \sinh y = -3$ —— (4)

$\Rightarrow \cos x = 0 \Rightarrow x = \pm(2n+1)\frac{\pi}{2}$

$\cosh y = 0 \Rightarrow \frac{e^y + e^{-y}}{2} = 0$ (impossible)

Putting $x = \frac{\pi}{2}$ in eq^n (4) $\sinh y = -3$

$y = \sinh^{-1}(-3)$

$z = x + iy = \pm(2n+1)\frac{\pi}{2} + i\sinh^{-1}(-3)$

LOGARITHM GENERAL POWER

$\ln z = \ln r + i\theta = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

Here $r = |z| = \sqrt{x^2+y^2}$ and $\theta = \arg z = \tan^{-1}\frac{y}{x}$

$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$, $n = 1, 2, 3 \ldots$

where $\mathrm{Ln}\, z$ = principal value of $\ln z$

where $-\pi < \arg z \le \pi$

No. 7 Find principal value of $\mathrm{Ln}(1 \pm i)$

Soln Here $1 \pm i = x + iy \Rightarrow x = 1, y = \pm 1$

$r = |z| = \sqrt{x^2+y^2} = \sqrt{2}$, $\theta = \tan^{-1}\frac{y}{x} = \pm\frac{\pi}{4}$

$\mathrm{Ln}\, z = \ln r + i\theta = \ln\sqrt{2} \pm i\,\pi/4$

No. 13 Find all the values of $\ln(-e^{-i})$

Soln $z = -e^{-i} = -[\cos 1 - i\sin 1] = x + iy$

$x = -\cos 1$ = real part, $y = \sin 1$ = Imag part

$r = |z| = \sqrt{x^2+y^2} = \sqrt{\cos^2 1 + \sin^2 1} = 1$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\sin 1}{-\cos 1}\right) = -\tan^{-1}(\tan 1)$

$= -1 \pm 2n\pi$

$\ln z = \ln r + i\theta = \ln 1 + i(-1 \pm 2n\pi)$

$= (-1 \pm 2n\pi)\, i$

No. 18 Solve $\ln z = e - \pi i$ for z

Soln $\ln z = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} = e - \pi i$

Equating real & imaginary parts

$\Rightarrow \ln\sqrt{x^2+y^2} = e$ and $\tan^{-1}\frac{y}{x} = -\pi$

$\Rightarrow \sqrt{x^2+y^2} = e^e$ and $\frac{y}{x} = \tan(-\pi) = 0$

$\Rightarrow x^2+y^2 = e^{2e}$ and $y = 0$

$\Rightarrow x^2 = (e^e)^2 \Rightarrow x = e^e$

$\Rightarrow z = x + iy = e^e$

No-23 Find the principal value of the ~~equation~~ expression $(1+i)^{-1+i}$

Sol^n Let $z = (1+i)^{-1+i}$

Then $\ln z = \ln (1+i)^{-1+i} = (-1+i)\ln(1+i)$

$= (-1+i)\left[\ln\sqrt{1^2+1^2} + i\tan^{-1}\left(\frac{1}{1}\right)\right]$

$= (-1+i)\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]$

$= -\ln\sqrt{2} - \frac{\pi}{4} + i\left[\ln\sqrt{2} - \frac{\pi}{4}\right]$

Hence $z = e^{-\ln\sqrt{2} - \frac{\pi}{4} + i\left[\ln\sqrt{2} - \frac{\pi}{4}\right]}$

$= e^{-\ln\sqrt{2}} \times e^{-\frac{\pi}{4}} \times e^{i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)}$

$= e^{\ln\frac{1}{\sqrt{2}}} \times e^{-\frac{\pi}{4}} \times e^{i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)}$

$= \frac{1}{\sqrt{2}} e^{-\pi/4}\left[\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right]$