LINE INTEGRAL IN COMPLEX PLANE

PARAMETRIC REPRESENTATION

(A). Line segment joining points z_1 & z_2

$$Z = z_1 + t(z_2 - z_1), \quad 0 \le t \le 1$$

(B) CIRCLE $|z - a| = r$ (Centre a, rad = r)

$$z - a = re^{it}, \quad 0 \le t \le 2\pi$$

(C) ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

put $\frac{x}{a} = \cos t$ $\quad \frac{y}{b} = \sin t$

$$z = x + iy = a\cos t + i b \sin t, \quad 0 \le t \le 1$$

(D) PARABOLA $y^2 = 4ax$

Let $x = at^2$ then $y = 2at$

$$z = x + iy = at^2 + 2at\,i$$

(E) UNIT CIRCLE $|z| = 1$ (Centre 0, rad = 1)

$$z = e^{it}, \quad 0 \le t \le 2\pi$$

(F) ANY CURVE $y = f(x)$

Let $x = t$ then $y = f(t)$

$$z = x + iy = t + i f(t)$$

No.2 Find parametric representation of line segment from $4+3i$ to $-5-i$

Sol $z_1 = 4 + 3i$, $z_2 = -5 - i$

$$z = z_1 + t(z_2 - z_1) = 4 + 3i + (-9 - 4i)t$$

$$0 \le t \le 1$$

No. 4 Find parametric representation of the ellipse $4x^2 + 9y^2 = 36$

Sol^n Given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ —— ①

Put $\frac{x}{3} = \cos t$, $\frac{y}{2} = \sin t$ so that given eq^n ① is $\cos^2 t + \sin^2 t = 1$ satisfied

$z = x + iy = 3\cos t + 2\sin t\, i, \quad 0 \le t \le 2\pi$

No. 15 Evaluate $\int_C \operatorname{Re} z\, dz$

C: shortest path from $1+i$ to $3+2i$

Sol^n $z_1 = 1+i, \quad z_2 = 3+2i$

parametric form of path C is

$z = z_1 + t(z_2 - z_1) = 1 + i + (2+i)t$

$\Rightarrow dz = (2+i)\,dt$ and $\operatorname{Re} z = 1+2t$, $\quad 0 \le t \le 1$

$\int_C \operatorname{Re} z\, dz = \int_0^1 (1+2t)(2+i)\,dt$

$= (2+i)\left[t + t^2\right]_0^1 = 2(2+i) = 4+2i$

No. 18. Evaluate $\int_C \bar{z}\, dz$, C: along parabola $y = x^2$ from 0 to $1+i$

Sol^n Let $x = t$ then $y = t^2$

$z = x + iy = t + it^2, \quad 0 \le t \le 1$

$\int_C \bar{z}\, dz = \int_0^1 (t - it^2)(1 + 2ti)\,dt$

$= \int_0^1 (t + 2t^3 - it^2 + 2t^2 i)\,dt$

$= \left[\frac{t^2}{2} + 2\frac{t^4}{4} - \frac{t^3}{3} i + \frac{2}{3} t^3 i\right]_0^1 = 1 + \frac{1}{3} i$

No. 19 Evaluate $\int_C \operatorname{Re} z^2\, dz$ C: Unit circle counter clockwise

Sol^n Parametric form of path C is

$z = re^{it}, \quad 0 \le t \le 2\pi \Rightarrow dz = i r e^{it}\, dt$

$\operatorname{Re} z^2$ = Real part of $r^2 e^{2it}$

= Real part of $r^2(\cos 2t + i \sin 2t) = r^2 \cos 2t$

$$I = \int_C \mathrm{Re}\, z^2 \, dz$$

$$= \int_0^{2\pi} r^2 \cos 2t \cdot i r e^{it} \, dt$$

$$= i r^3 \int_0^{2\pi} e^{it} \cos 2t \, dt$$

<u>Formula</u> $\int e^{at} \cos bt \, dt$

$$= \frac{e^{at}}{a^2+b^2} (a \cos bt + b \sin bt) + c$$

$$= i r^3 \left[\frac{e^{it}}{-1+4} (i \cos 2t + 2 \sin 2t) \right]_0^{2\pi}$$

$$= i r^3 \left[\frac{e^{2\pi i}}{3} (i + 0) - \frac{1}{3} (i + 0) \right]$$

$$= \frac{i r^3}{3} \left(e^{2\pi i} - 1 \right)$$

$$= \frac{i r^3}{3} \left(\cos 2\pi + i \sin 2\pi - 1 \right).$$

$$= \frac{i r^3}{3} (1 + 0 - 1) = 0$$

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D then for every simple closed path C in D, $\oint_C f(z)\,dz = 0$

EX-1 No singularities (Entire Function)

$\oint_C e^z dz = 0$ (Since e^z is analytic every where for any closed path)

$\oint \cos z\,dz = 0$ $\quad \oint_C z^n dz = 0$

EX-2 Singularity outside contour

$\oint_C \sec z\,dz = 0$ where $C: |z| = 1$

$\oint \frac{dz}{z^2+4} = 0$ $\quad C: |z| = 1$

EX-3 Non-analytic Function

$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it}\, ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$

where $C: |z| = 1$

EX-4 Analytic sufficient, not necessary

$\oint \frac{dz}{z^2} = 0$ $\quad C: |z| = 1$

Does not follow Cauchy Integral theorem as $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$

EX-5 Simple connected Essential

$\oint \frac{dz}{z} = 2\pi i$ ~~around circle $C: |z| = 1$ counter clockwise~~

C lies in annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $\frac{1}{z}$ is analytic but domain is not simply connected.

Cauchy's Integral theorem can not be applied as domain is not simply connected.

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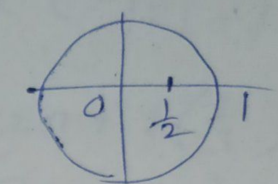
Cauchy's Integral theorem can not be applied as domain is not simply connected.

No. 7 $\quad I = \oint_C e^{-z^2}\,dz$, $C: |z| = 1$ Counter-clockwise

Sol^n $\quad f(z) = e^{-z^2} = \dfrac{1}{e^{z^2}}$ is analytic everywhere and domain is unit circle $|z| = 1$ which is simply connected. Hence by Cauchy's Integral theorem $I = 0$

No. 11 $\quad \oint \dfrac{dz}{2z-1}$ $\quad C: |z| = 1$ counter-clockwise



$C: |z| = 1$
Circle with Centre 0, rad 1

Sol^n $\quad f(z) = \dfrac{1}{2z-1}$ which is not analytic at $2z - 1 = 0$
$z = \frac{1}{2}$ which lies inside $C: |z| = 1$
Hence $f(z)$ is not analytic inside $C: |z| = 1$ and we can not apply Cauchy's Integral Th^m.

No. 22 $\quad I = \oint_C \dfrac{2z-1}{z^2 - z}\,dz = \oint_C \dfrac{z + (z-1)}{z(z-1)}\,dz$

$= \oint_C \dfrac{1}{z-1}\,dz + \int_C \dfrac{dz}{z} = I_1 + I_2$ (Say) $\quad$ C is as in fig. 1

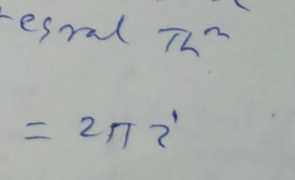


Fig-1

Both $z = 0, 1$ where $f(z)$ is not analytic lie inside C. Hence Cauchy's Integral Th^m can not be applied

$I_1 = \int_C \dfrac{dz}{z-1} = \int_0^{2\pi} \dfrac{i e^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$

(Put $z - 1 = e^{it}$, $0 \le t \le 2\pi$, $\Rightarrow dz = i e^{it}\,dt$)

$I_2 = \int_C \dfrac{dz}{z} = \int_0^{2\pi} \dfrac{i e^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$

$I = I_1 + I_2 = 4\pi i$

Cauchy's Integral Formula: Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simply closed path C in D that encloses z_0 $\quad \int_C \dfrac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0)$

No. 8 $I = \oint \frac{z^2 \sin z}{3z-1} dz$, $C: |z| = 1$ counterclockwise

Soln $I = \oint \frac{z^2 \sin z}{3} \frac{dz}{(z - \frac{1}{3})} = \oint_C f(z) \frac{dz}{z - z_0}$

Here $f(z) = \frac{z^2}{3} \sin z$ and $z - z_0 = z - \frac{1}{3}$
$z_0 = \frac{1}{3}$

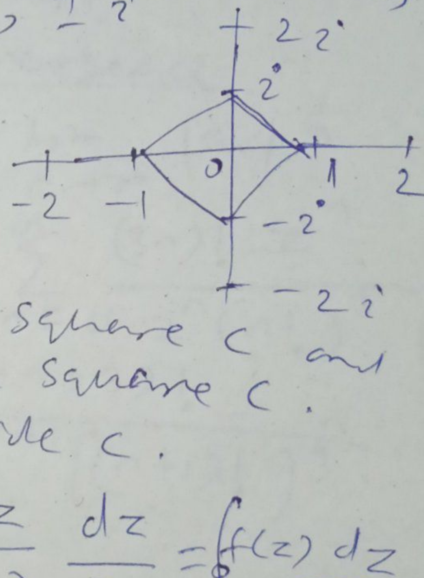
$z_0 = \frac{1}{3}$ lies inside $C: |z| = 1$, circle with centre 0, rad 1.
Hence $f(z)$ is not analytic inside $C: |z| = 1$
By Cauchy's Integral formula

$$I = \oint \frac{f(z)\, dz}{z - z_0} = 2\pi i f(z_0) = 2\pi i f\left(\frac{1}{3}\right)$$

$$= 2\pi i\, f(z)\Big]_{z=\frac{1}{3}} = 2\pi i \left.\frac{z^2}{3} \sin z\right]_{z=\frac{1}{3}} = \frac{2\pi i}{27} \sin \frac{1}{3}$$

No. 12 $\oint \frac{4 - \sin z}{z^2 - 2z} dz$, C: square with vertices $\pm 1, \pm i$

Soln Integrand $\frac{4 - \sin z}{z^2 - 2z}$ is not analytic at $z(z-2) = 0$
$\Rightarrow z = 0, 2$



The point $z = 0$ lies inside square C and
$z = 2$ lies outside square C.
Hence $f(z)$ is not analytic inside C.

$$I = \oint \frac{4 - \sin z}{z^2 - 2z} dz = \oint_C \frac{4 - \sin z}{(z-2)} \frac{dz}{z} = \oint_C \frac{f(z)\, dz}{z - z_0}$$

Here $f(z) = \frac{4 - \sin z}{z - 2}$ and $z - z_0 = z \Rightarrow z_0 = 0$
By Cauchy Integral formula $I = 2\pi i f(z_0)$

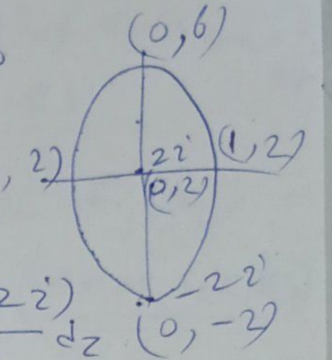
$$= 2\pi i\, f(0) = 2\pi i \left[\frac{4 - \sin z}{z - 2}\right]_{z=0} = -4\pi i$$

No. 11 $\oint_C \frac{dz}{z^2 + 4}$ C: Ellipse $4x^2 + (y-2)^2 = 16$

$$\frac{x^2}{2^2} + \frac{(y-2)^2}{4^2} = 1$$

$\frac{1}{z^2+4} \Rightarrow$ not analytic at $z^2+4=0$

$z = \pm 2i$ which lies inside the given ellipse $C: 4x^2+(y-2)^2=16$



$$I = \oint_C \frac{dz}{z^2+4} = \oint_C \frac{\frac{1}{4i}\left[(z+2i)-(z-2i)\right]}{(z+2i)(z-2i)}\,dz$$

$$= \frac{1}{4i}\oint \frac{1}{z-2i}\,dz - \frac{1}{4i}\int \frac{1}{z+2i}\,dz = \frac{I_1}{4i} - \frac{I_2}{4i}$$

$$I_1 = \frac{1}{4i}\oint 1 \cdot \frac{dz}{z-2i} = 2\pi i f(z_0) = 2\pi i\, f(2i) = 2\pi i\,[1]_{z=2i} = 2\pi i$$

$$I_2 = \frac{1}{4i}\int 1 \cdot \frac{dz}{z+2i} = 2\pi i f(-2i) = 2\pi i\,[1]_{z=-2i} = 2\pi i$$

$$I = \frac{I_1}{4i} - \frac{I_2}{4i} = \frac{2\pi i}{4i} - \frac{2\pi i}{4i} = 0$$

Power series, Radius of Convergence

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \quad \text{and} \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

No. 11

Find radius of convergence of $\sum_{n=0}^{\infty} \frac{(3n)!\, z^n}{2^n (n!)^3}$

Sol^n $a_n = \frac{(3n)!}{2^n (n!)^3}$ $\quad a_{n+1} = \frac{(3n+3)!}{2^{n+1}\left[(n+1)!\right]^3}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)(3n)!}{2^n \times 2\left[(n+1)n!\right]^3} \times \frac{2^n (n!)^3}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{2(n+1)(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{3(n+1)}{n+1}\,\frac{(3n+2)(3n+1)}{(n+1)(n+1)} = \frac{3}{2}\lim_{n\to\infty}\frac{(3+\frac{2}{n})(3+\frac{1}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})}$$

$$= \frac{3}{2}\,\frac{3\times 3}{1\times 1} = \frac{27}{2} \quad \text{Hence } R = \frac{2}{27}$$

Centre of convergence $= 0$, Rad of convergence $= \frac{2}{27}$

TAYLOR SERIES & MACLAURIN SERIES

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0)$$
$$+ \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n \quad \text{(TAYLOR SERIES)}$$

No. 26 Find Taylor series of $f(z) = \cos^2 z$, $z_0 = \pi$

Sol^n $f(z) = \cos^2 z \Rightarrow f(z_0) = f(\pi) = 1$

$f'(z) = 2\cos z(-\sin z) = -\sin 2z \Rightarrow f'(z_0) = 0$

$f''(z) = -2\cos 2z \Rightarrow f''(z_0) = -2$

$f'''(z) = 4\sin 2z \Rightarrow f'''(z_0) = 0$

$f^{iv}(z) = 4\cos 2z \Rightarrow f^{iv}(z_0) = 4$

$$f(z) = f(z_0) + (z - \pi) f'(z_0) + \frac{(z-\pi)^2}{2!} f''(z_0) + \cdots$$
$$= 1 + 0 + \frac{(z-\pi)^2}{2!}(-2) + 0 + \frac{(z-\pi)^4}{4!} 4 + \cdots$$

Maclaurin series

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$$

No. 3 Find Maclaurin series of $f(z) = \sin z$

Sol^n $f(z) = \sin z \Rightarrow f(0) = 0$

$f'(z) = \cos z \Rightarrow f'(0) = 1$

$f''(z) = -\sin z \Rightarrow f''(0) = 0$

$f'''(z) = -\cos z \Rightarrow f'''(0) = -1$

$f^{iv}(z) = \sin z \Rightarrow f^{iv}(0) = 0$

$f^{v}(z) = \cos z \Rightarrow f^{v}(0) = 1$

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$
$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

No. 9 Find Maclaurin series of $\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ by termwise integration.

Sol $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$e^{-z^2} = 1 - z^2 + \frac{z^4}{2!} - \frac{z^6}{3!} + \cdots$

$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \int_0^z \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt$

$= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3} + \frac{t^5}{5(2!)} - \frac{t^7}{7(3!)} + \cdots \right]_0^z$

$= \frac{2}{\sqrt{\pi}} \left[z - \frac{z^3}{3} + \frac{z^5}{5(2!)} - \frac{z^7}{7(3!)} + \cdots \right]$

Important Special Maclaurin series

$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n$

$(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, $(1+z)^{-m} = \sum_{n=0}^{\infty} {}^{-m}C_n z^n$

$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}$

LAURENT SERIES

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

principal part of Laurent series (Negative powers)

No. 9 Find Laurent series of $\frac{1}{z^2+1}$, $z_0 = i$ that converges for $0 < |z-z_0| < R$

Sol $f(z) = \frac{1}{z^2+1}, \quad z_0 = i$

We want to expand in powers of $z - z_0$ i.e. in powers of $z - i$

$$f(z) = \frac{1}{z^2+1} = \frac{1}{z^2-(-1)} = \frac{1}{z^2-i^2}$$

$$= \frac{1}{(z-i)}\frac{1}{(z+i)} = \frac{1}{(z-i)}\frac{1}{(z-i+2i)}$$

$$= \frac{1}{(z-i)}\frac{1}{2i\left[1+\frac{z-i}{2i}\right]} = \frac{1}{2i(z-i)}\left[1+\frac{z-i}{2i}\right]^{-1}$$

$$= \frac{1}{2i(z-i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

Using Maclaurin series $(1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$

which converges for $\left|\frac{z-i}{2i}\right| < 1$

$$\frac{|z-i|}{|2i|} < 1 \Rightarrow \frac{|z-i|}{2} < 1$$

$$0 < |z-i| < 2$$

As a geometric series $\sum a r^n$ converges for $|r| < 1$

No. 18 Develop $f(z) = \frac{2z-3i}{z^2-3iz-2}$ in Laurent series valid for $1 < |z| < 2$

Sol $f(z) = \frac{(z-2i)+z-i}{(z-2i)(z-i)} = \frac{1}{z-i} + \frac{1}{z-2i}$

$$= \frac{1}{z(\cancel{\#}-\frac{i}{z})} + \frac{1}{-2i\left(1-\frac{z}{2i}\right)}$$

$\cancel{= \frac{1}{z}(z-i)^{-1}}$

$$= \frac{1}{z}\cdot\left(1-\frac{i}{z}\right)^{-1} - \frac{1}{2i}\left(1-\frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n - \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n$$

which converges for $\left|\frac{i}{z}\right| < 1$ and $\left|\frac{z}{2i}\right| < 1$

$\frac{|i|}{|z|} < 1$ and $\frac{|z|}{|2i|} < 1$

$\frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1 \Rightarrow |z| > 1$ and $|z| < 2$

$1 < |z| < 2$ is the valid region of convergence of these Laurent series

Singularity, zero

A function $f(z)$ has singularity at $z = z_0$ if $f(z)$ is not analytic at z_0.

(A) $z = z_0$ is called isolated singularity of $f(z)$ if $z = z_0$ has a neighbourhood without further singularities of $f(z)$

Ex: $\tan z$ at $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

$\tan \frac{1}{z}$ has non-isolated singularity at $z = 0$

(B) If principal part of Laurent series $\sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$ has only finitely many terms then poles of order m at $z = z_0$

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

(C) If principal part has infinitely many terms $z = z_0$ is an isolated essential singularity

Ex: $e^{1/z}$ at $z = 0$

(D) If $\lim_{z \to a} (z - a) f(z)$ exists then removable singularity at $z = a$

Ex $\frac{\sin z}{z}$ at $z = 0$ because $\lim_{z \to 0} (z - 0) \frac{\sin z}{z} = 0$ exists

No. 2 Find zeros of $(z^4 - 1)^4$

Sol $(z^4 - 1)^4 = 0$

$z^4 = 1 \Rightarrow z = 1, -1, i, -i$ (order 4)

No. 13 Find location & type of singularity of $f(z) = \frac{2}{z^3} - \frac{1}{z}$

Sol^n pole of order 3 at $z = 0$

No. 11 Find location & type of singularity of $f(z) = \tan \frac{\pi}{2} z = \frac{\sin \frac{\pi}{2} z}{\cos \frac{\pi}{2} z}$

Solution Singularity (not analytic) at $\cos \frac{\pi z}{2} = 0$

~~$z = \pm 1$~~ $z = \pm 1, \pm 3, \pm 5 \ldots$ (simple poles)

$f(\frac{1}{z}) =$ ~~cot~~ $\tan\left(\frac{\pi}{2z}\right)$

At $z = \infty$ $f(\frac{1}{z}) = \frac{\pi}{2z} + \frac{1}{3}\left(\frac{\pi}{2z}\right)^3 + \frac{2}{15}\left(\frac{\pi}{2z}\right)^5 + \ldots$

which is singular at $z = 0$

Essential singularity at $z = \infty$

Since $\tan z = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \ldots$

CAUCHY RESIDUE THEOREM & APPLICATIONS

Let $f(z)$ be analytic inside a simple closed path C and on C except for finitely many singular points $z_1, z_2, \ldots, z_k$ inside C, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

FORMULA TO FIND RESIDUE

For simple poles $f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z) = \left.\frac{p(z)}{q'(z)}\right]_{z=z_0}$$

Pole of order $m > 1$

$$f(z) = \frac{b_m}{(z - z_0)^m} + \ldots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]\right]$$

No. 17 $I = \oint \frac{z+1}{z^4 - 2z^3} dz \quad C: |z| = \frac{1}{2}$

Evaluate using residue Th^m

Sol poles at $z^4 - 2z^3 = 0 \Rightarrow z^3(z-2) = 0$

$z = 2$ (order 1) $\quad z = 0$ (order 3)

Only pole $z = 0$ (order $m = 3$) lies inside $C: |z| = \frac{1}{2}$

$$\underset{z=0}{\text{Res}} f(z) = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$= \frac{1}{2} \lim_{z \to 0}\left[\frac{d^2}{dz^2}\left(\frac{z+1}{z-2}\right)\right] = -\frac{3}{8}$$

By Cauchy Residue Th^m

$I = 2\pi i \times$ Sum of residues (evaluated at points which lie inside C)

$= 2\pi i\left(-\frac{3}{8}\right) = -\frac{3}{4}\pi i$

No. 16 $I = \oint \frac{e^z + z}{z^3 - z} dz, \quad C: |z| = \frac{\pi}{2}$

Evaluate using Residue Th^m

Sol^n poles at $z^3 - z = 0 \Rightarrow z(z-1)(z+1) = 0$

All poles $z = 0, 1, -1$ (simple poles)

lie inside $C: |z| = \frac{\pi}{2} = 1.571$

$$\underset{z=0}{\text{Res}} f(z) = \frac{p(z)}{q'(z)} = \left.\frac{e^z + z}{3z^2 - 1}\right]_{z=0} = -1$$

$$\underset{z=1}{\text{Res}} f(z) = \left.\frac{e^z + z}{3z^2 - 1}\right]_{z=1} = \frac{1+e}{2}$$

$$\underset{z=-1}{\text{Res}} f(z) = \left.\frac{e^z + z}{3z^2 - 1}\right]_{z=-1} = \frac{e^{-1} - 1}{2} = \frac{1-e}{2e}$$

By Cauchy Residue theorem

$I = 2\pi i \times$ Sum of residues (evaluated at points which lies inside C)

$$= 2\pi i\left[-1 + \frac{1+e}{2} + \frac{1-e}{2e}\right]$$

EVALUATION OF REAL INTEGRALS

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

Put $e^{i\theta} = z \Rightarrow dz = ie^{i\theta} d\theta = iz\, d\theta$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

No. 3 Evaluate $\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\, d\theta$ using residue theorem

Sol^n^ Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$I = \int_C \frac{1 + \frac{1}{2i}\left(z - \frac{1}{z}\right)}{3 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} \qquad C: |z| = 1$$

$$= \int_C -\frac{(2iz + z^2 - 1)}{z(6z + z^2 + 1)} dz \quad \left(\text{multiply } 2iz \text{ in Num \& denom}\right)$$

$$f(z) = \frac{-(2iz + z^2 - 1)}{z(6z + z^2 + 1)} \text{ has}$$

poles at $z = 0$, $z^2 + 6z + 1 = 0$

$$z = -3 \pm 2\sqrt{2}$$

Only poles $z = 0$, $z = -3 + 2\sqrt{2} = -0.172$ lie inside $C: |z| = 1$ (poles of order 1)

$$\operatorname*{Res}_{z=-3+2\sqrt{2}} f(z) = \left.\frac{p(z)}{q'(z)}\right]_{z=-0.172} = \left.\frac{-(2iz + z^2 - 1)}{12z + 3z^2 + 1}\right]_{z=-0.172}$$

$$= -1 + 0.383i$$

$$\operatorname*{Res}_{z=0} f(z) = \left.\frac{-(2iz + z^2 - 1)}{12z + 3z^2 + 1}\right]_{z=0} = 1$$

By Cauchy Residue theorem

$I = 2\pi i \times$ sum of residues

$= 2\pi i\,[-1 + 0.383\,i + 1] = -0.766\pi$

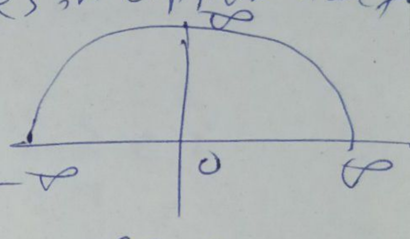
No. 9 Evaluate the improper integral $I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ using residues.

Sol^n Consider $I = \int_{-\infty}^{\infty} \frac{dz}{(1+z^2)^2}$

$f(z) = \frac{1}{(1+z^2)^2}$ has poles at $1 + z^2 = 0$

$z = i, -i$ (order 2)

Only pole $z = i$ (order 2) lies in upper half plane (or contour)



$-\infty < x < \infty$

$0 < y < \infty$

$\underset{z=i}{\operatorname{Res}} f(z) = \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right]$

$= \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \frac{1}{(1+z^2)^2}\right]$

$= \lim_{z \to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] = \lim_{z \to i} \frac{-2}{(z+i)^3}$

$= -\frac{2}{8i^3} = \frac{1}{4i}$

$I = 2\pi i \times$ sum of residues $= 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$

No. 20 $I = \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2} = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{2iz}}{(z^2+1)^2}$

$=$ Real part of I_1 (say)

Sol^n $f(z) = \frac{e^{2iz}}{(z^2+1)^2}$

poles at $(z^2+1) = 0$, $z = i, -i$ (order 2)

only pole $z = i$ lies inside contour

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{e^{2iz}}{(z^2+1)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz} \frac{e^{2iz}}{(z+i)^2}$$

$$= \lim_{z\to i} \frac{2i e^{2iz}(z+i)^2 - 2e^{2iz}(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{2e^{2iz}(-1+iz-1)(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{(-4+2iz)e^{2iz}}{(z+i)^3} = -\frac{3}{4}e^{-2}i$$

$I_1 = 2\pi i \times$ sum of residues

$$= 2\pi i\left(-\frac{3}{4}e^{-2}i\right) = \frac{3\pi}{2}e^{-2}$$

$$I = \text{Real part of} \int_{-\infty}^{\infty} \frac{e^{2iz}}{(z^2+1)^2}$$

$$= \text{Real part of } I_1 = \frac{3\pi}{2}e^{-2}$$

Similarly $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2x}{(x^2+1)^2}$

$= \text{Imaginary part of} \displaystyle\int_{-\infty}^{\infty} \frac{e^{2iz}}{(z^2+1)^2}$

$= 0$