

Q (2, $\geq$), check for partially ordered relation.

Soln Reflexive

$$a \geq a \quad \forall a \in \mathbb{Z}$$

$\geq$ - reflexive

Anti-Symmetric

$$a \geq b, b \geq a, \forall a, b \in \mathbb{Z}$$

$$\Rightarrow a = b$$

$\geq$ - anti-symmetric

Transitive

$$a \geq b, b \geq c \quad \forall a, b, c \in \mathbb{Z}$$

$$\Rightarrow a \geq c$$

$\geq$ is transitive

So, it is partially ordered Relation

Q Show that the following divides relation are partially ordered on A:

i) $m|n$, m divides n where $A = \mathbb{N}$

ii) $m|n$, m divides n where

A = the set of +ve integers divisors of 36.

Check if it is partially ordered Relation or not?

Soln $n|n \rightarrow$ hence the relation is reflexive

$$\rightarrow \exists m|n \text{ and } n|m$$

$$\Rightarrow m = n \rightarrow \text{Anti-Symmetric}$$

$$\rightarrow \exists m|n \text{ and } n|p$$

$$\Rightarrow m|p \text{ (transitive)}$$

Hence, it is partially ordered Relation.

Symmetric
 $R \cup R^{-1}$

$$R^{-1} = \{(2,1), (3,2), (1,3)\}$$

$$R \cup R^{-1} = \{(1,2), (2,2), (3,1), 0, (2,1), (3,2), (1,3)\}$$

Transitive

$$R = R \cup R^2 \cup R^3$$

$$R^2 = R \circ R = \{(1,3), (2,1), (3,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1), (2,2), (3,3)\}$$

$$R = \{(1,2), (2,3), (3,1), (1,1), (2,2), (3,3), (1,3), (2,1), (3,2)\}$$

Posets (partially ordered set)

A partially ordered set or a poset is pair

$P = (X, \leq)$ where X is a non-empty set.
 $\leq$ is a partial order in X .

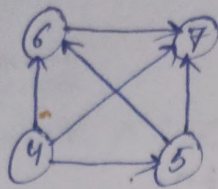
i.e. for $x, y \in X$

1. $x \leq x$ Reflexive
2. $x \leq y, y \leq x$ implies $x = y$ (Anti-symmetric)
3. $x \leq y, y \leq z$ implies $x \leq z$ (transitive)

↳ partial ordered set.

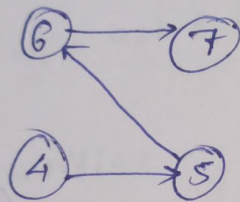
→ A Relation R is said to be partially ordered relation if it is reflexive, anti-symmetric and transitive.

Step 2 discard all same group,

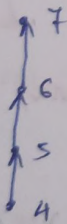


Step 3 Remove Transitive edges,

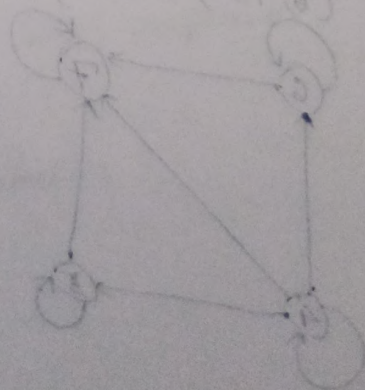
$(4,5), (5,6)$ $(5,6), (6,7)$
 ~~$(4,6)$~~ ~~$(5,7)$~~
 Transitive Relⁿ



Step 4 & 5

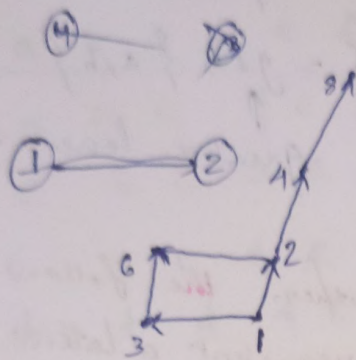


(Hasse Diagram)



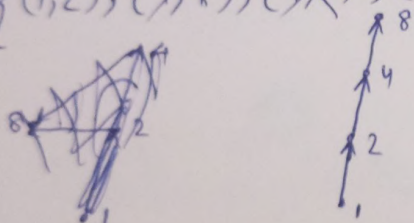
Ex Let $A = \{1, 2, 3, 4, 6, 8\}$ be ordered by the relation 'a divides b'.

Soln $R = \{(1, 2), (1, 3), (1, 4), (1, 6), (1, 8), (2, 4), (2, 6), (2, 8), (3, 6), (4, 8)\}$



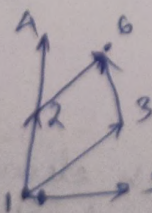
Ex $D_8 = \{1, 2, 4, 8\}$. The relation is divisibility.

$\Rightarrow R = \{(1, 2), (1, 4), (1, 8), (2, 4), (2, 8), (4, 8)\}$



Ex $D = \{1, 2, 3, 4, 5, 6\}$. The relation is divisibility.

$\Rightarrow R = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)\}$



Lattice :-

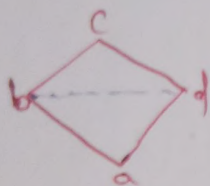
A poset is a lattice if $\forall a, b \in L$

$\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L .

① $a \vee b = a \text{ join } b = \sup\{a, b\} = \text{lub}\{a, b\}$
 $\downarrow$
 least upward bound.

② $a \wedge b = a \text{ meet } b = \inf\{a, b\} = \text{glb}\{a, b\}$
 $\uparrow$
 greatest lower bound.

Ex



Determine whether the following Hasse diagrams represent a lattice or not.

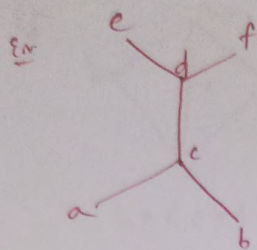
v	lub			
	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	d	d	d

Since each subset of two element has least upper bound and greatest lower bound. So,

L containing lub.

v	glb			
	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since each element of 2 elements has least upper bound & greatest lower bound L becomes a lattice.



Ans

b

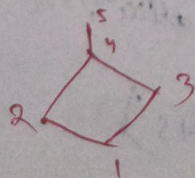
V	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	-
f	f	f	f	f	f	f

Ans

a	a	b	c	d	e	f
a	a	-	a	a	a	a
b	-	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

Assignment

Q Determine whether the following diagram represent a lattice or not.



Properties of lattice and distributive lattice

Let $(L, \leq)$ be a lattice & $a, b \in L$

① Idempotent law: $a \in L$

$$a \vee a = a$$

$$a \wedge a = a$$

② Commutative law: $\forall a, b \in L$

$$a \vee b = b \vee a$$

$$a \wedge b = b \wedge a$$

③ Associative law: $\forall a, b, c \in L$

$$(a \vee b) \vee c = a \vee (b \vee c)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

④ Absorption law: for $a, b \in L$

① $a \vee (a \wedge b) = a$

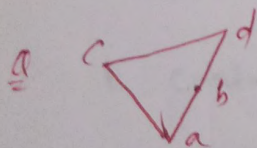
② $a \wedge (a \vee b) = a$

⑤ Distributive lattice

A lattice $(L, \leq)$ is called a distributive lattice if for any $a, b, c \in L$

① $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

② $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$



① LHS

$$a \vee (b \wedge c) = a \vee a = a$$

$$(a \vee b) \wedge (a \vee c) = b \wedge c = a$$

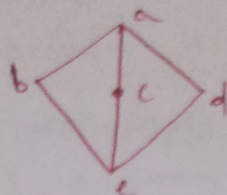
LHS = RHS.

Hence, distributive lattice.

② LHS =

$$a \wedge (b \vee c) = a \wedge b \vee (a \wedge c) = a \quad (\text{LHS} = \text{RHS})$$

Ex



$$b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d) \quad \text{check for distributive lattice.}$$

Soln LHS

$$b \wedge a = b$$

RHS

$$(b \wedge c) \vee (b \wedge d)$$

$$= e \vee e$$

$$= e$$

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Bounded lattice :-

A lattice $(L, \leq)$ is called bounded lattice if it has greatest element 1 and least element 0. or L has greatest and least element in L .

→ $(L, \leq)$ is bounded then for any element $a \in L$

$$a \vee 1 = 1$$

$$a \vee 0 = a$$

$$a \wedge 1 = a$$

$$a \wedge 0 = 0$$

→ Every finite lattice is bounded.

Complement of lattice :-

complement of an element in a lattice $(L, \leq)$ in bounded lattice with least element '0' and greatest element 1.

$a \in L, a' \in L$ is called the complement of a

$$a \vee a' = 1, a \wedge a' = 0$$

→ It is not necessary that every element has a complement.

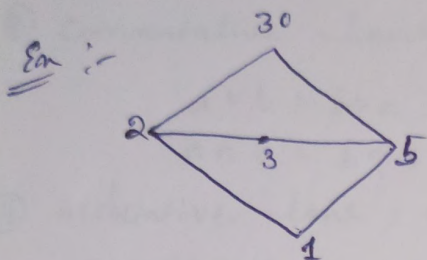
→ An element $a \in L$ have more than one complement.

Complimented Lattice

A lattice $(L, \leq)$ is called complemented

i) L is bound

ii) Every element $a \in L$ has a complement.



$$A = \{1, 2, 3, 5, 30\}$$

And $a \leq b$ if a divides b
Then find complement of 2.

Sol

$$2 \vee 2' = 30$$

$$2 \wedge 2' = 1$$

$$2 \vee 3 = 30$$

$$2 \wedge 3 = 1$$

$$glb = 1$$

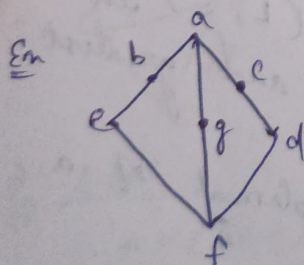
$$lub = 30$$

So, complement of 2 is 3.

$$5 \vee 3 = 30$$

$$5 \wedge 3 = 1$$

So, complement of 5 is 3.



If a and b are elements in a distributive lattice L and if a' complement of a . Then,

$$i) (a')' = a$$

$$ii) (a \vee b)' = a' \wedge b'$$

$$iii) (a \wedge b)' = a' \vee b'$$

$$iv) a \vee (a' \wedge b) = a \vee b$$

$$v) a \wedge (a' \vee b) = a \wedge b$$

Boolean Algebra

A non-empty set B with two binary operations $(+)$ and $(\cdot)$, a unary operation $(')$ and two distinct elements 0 & 1 is called a Boolean Algebra denoted by $(B, +, \cdot; 0, 1)$ iff the binary properties are satisfied.

Axioms of Boolean Algebra

If $a, b, c \in B$, Then

① Commutative law

① a $a + b = b + a$

② $a \cdot b = b \cdot a$

② Distributive law

① $a + (b \cdot c) = (a + b) \cdot (a + c)$

② $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

③ Identity law

① $a + 0 = a$ ② $a \cdot 1 = a$

④ Complement laws

① $a + a' = 1$

② $a \cdot a' = 0$

Some Basic Results of Boolean Algebra

Let $a, b, c \in B$. Then,

① Idempotent law

① $a + a = a$ | ② $a \cdot a = a$

$\Rightarrow a = a + 0$ (identity rule)

$= a + (a \cdot a')$ (complement rule)

$= (a + a) \cdot (a + a')$ (distributive law)

$= (a + a) \cdot 1$ (complement)

$= a + a$

$\Rightarrow a \cdot a = a$

$a \cdot a = a \cdot a + 0$ (identity)

$= a \cdot a + a \cdot a'$ (complement)

$= a \cdot (a + a')$ (distributive)

$= a \cdot 1$ (complement)

$= a$

⑥ Boundness Law

① $a + 1 = 1$

$$\begin{aligned} \Rightarrow 1 &= a + a' \text{ (Complement)} \\ &= a + a' \cdot 1 \text{ (Identity)} \\ &= (a + a') \cdot (a + 1) \text{ (Distributive)} \\ &= 1 \cdot (a + 1) \text{ (Complement)} \\ &= (a + 1) \cdot 1 \text{ (Commutative)} \\ &= a + 1 \text{ (Identity rule)} \end{aligned}$$

② $a \cdot 0 = 0$

$$\begin{aligned} \Rightarrow 0 &= a \cdot a' \text{ (Complement)} \\ &= a \cdot (a' + 0) \text{ (Identity)} \\ &= a \cdot a' + a \cdot 0 \text{ (Distributive)} \\ &= 0 + a \cdot 0 \text{ (Comp)} \\ &= a \cdot 0 + 0 \text{ (Commutative)} \\ &= a \cdot 0 \end{aligned}$$

③ Absorption Law

Let $a, b \in B$

① $a + (a \cdot b) = a$

$$\begin{aligned} \Rightarrow (a \cdot 1) + (a \cdot b) &\text{ {Identity}} \\ \Rightarrow a \cdot (1 + b) &\text{ {Distributive}} \\ \Rightarrow a \cdot (b + 1) &\text{ {Commutative}} \\ \Rightarrow a \cdot 1 &\text{ { } } \{ b + 1 = 1 \text{ by} \\ &\text{boundness law} \} \\ \Rightarrow a \end{aligned}$$

② $a \cdot (a + b) = a$

$$\begin{aligned} \Rightarrow a \cdot (a + b) &\in \text{ } \\ \Rightarrow (a \cdot 1) \cdot (a + b) &\in \text{ } \\ \Rightarrow \text{ } &\{ 1 \cdot b = 0 \} \\ \Rightarrow a \end{aligned}$$

Recurrence Relation

A sequence can be defined by giving a general formula for the n^{th} term or by writing a few of its terms. Such a relation is called a recurrence relation or difference equation.

Linear recurrence relation

$$C_0 a_n + C_1 a_{n-1} + \dots + C_k a_{n-k} = f(n)$$

C_i 's are constants.

↳ order K (i.e., $n - n + K$)

↳ degree 1 (highest power of an recurrence relⁿ)

$f(n) = 0 \rightarrow$ It is a homogenous recurrence relation.

$f(n) \neq 0 \rightarrow$ It is a non-homogenous recurrence relation.

Solution of Recurrence Relation :-

- i) By characteristic method
- ii) By generating function.

Homogenous Recurrence Relation :

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} \quad \text{--- (1)}$$

Let $a_n = r^n$

$$r^n = C_1 r^{n-1} + C_2 r^{n-2} + \dots + C_k r^{n-k}$$

Dividing r^{n-k} throughout the equation

$$r^k - C_1 r^{k-1} - C_2 r^{k-2} - \dots - C_k = 0 \quad \text{--- (2)}$$

Characteristics eqⁿ of homogenous recurrence solⁿ.

Case I (Distinct Root)

Let $r_1, r_2, \dots, r_k$ be distinct roots.

Then the solⁿ

$$a_n^{(n)} = b_1 r_1^n + b_2 r_2^n + \dots + b_k r_k^n$$

Where $b_1, b_2, \dots, b_k$ are arbitrary constants.

Case II (Repeated Roots)

$$a_n^{(n)} = (b_1 + b_2 n + b_3 n^2 + \dots + b_k n^{k-1}) r^n$$

Case III (Complex root)

$$a_n = r^n (A \cos n\theta + B \sin n\theta)$$

(Where A and B are constants)

Ex $a_n + 5a_{n-1} + 6a_{n-2} = 0$

Solⁿ let $a_n = r^n$
 $a_{n-1} = r^{n-1}$
 $a_{n-2} = r^{n-2}$

$r^n + 5r^{n-1} + 6r^{n-2} = 0$
 $\rightarrow$ Char. eqⁿ.

Dividing r^{n-2} throughout the eqⁿ,

$r^2 + 5r + 6 = 0$
 $r^2 + 2r + 3r + 6 = 0$
 $r(r+2) + 3(r+2) = 0$
 $(r+2)(r+3) = 0$

$\Rightarrow r = -2, -3$ (distinct real roots)

$a_n^{(n)} = C_1(-2)^n + C_2(-3)^n$

Ex $a_n = 8a_{n-1} + 16a_{n-2} = 0$

Solⁿ $a_2 = 6, a_3 = 80$

Let $a_n = r^n$
 $r^n - 8r^{n-1} + 16r^{n-2} = 0$

$\Rightarrow r^2 - 8r + 16 = 0$

$(r-4)^2 = 0$

$r = 4, 4$ (Repeated real roots)

$a_n^{(n)} = (C_1 + C_2 n) 4^n$

$a_2 = (C_1 + 2C_2) 4^2 = 6$

$16C_1 + 32C_2 = 6 \quad \text{--- (1)}$

$a_3 = (C_1 + C_2 \times 3) 4^3 = 80$

$64C_1 + 192C_2 = 80 \quad \text{--- (2)}$

Solving ① & ② ,

$$64C_1 + 192C_2 = 80$$

$$4 \times 64C_1 + 128C_2 = 24$$

$$\underline{64C_2 = 56}$$

$$C_2 = \frac{56}{64}$$

$$C_2 = 0.875$$

$$64C_1 = -168 + 80$$

$$C_1 = \frac{-88}{64}$$

$$C_1 = -1.375$$

$$a_n^{(n)} = (-1.375 + 0.875n) 4^n$$

Eqn $a_n - 4a_{n-1} + 5a_{n-2} = 0$

Soln Let $a_n = r^n$

$$r^n - 4r^{n-1} + 5r^{n-2} = 0$$

$$\Rightarrow r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm 2i}{2} = 2 \pm i$$

$$|r| = \sqrt{2^2 + 1^2} = \sqrt{5}, \theta = \tan^{-1} y/x = \tan^{-1} 1/2$$

$$a_n = r^n (C_1 \cos n\theta + C_2 \sin n\theta)$$

$$\text{where } |r| = \sqrt{5}, \theta = \tan^{-1} 1/2$$

Ex $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

Sol let $a_n = r^n$

$$r^n - 8r^{n-1} + 21r^{n-2} - 18r^{n-3} = 0$$

Ans $r = 2, 3, 3$

$$a_n^{(h)} = C_0 2^n + (C_1 + C_2 n) 3^n$$

C_0, C_1, C_2 are arbitrary const.

Assignment

Ex: ① $a_n = 4(a_{n-1} - a_{n-2})$

$$a_0 = a_1 = 1$$

② solve $a_n = a_{n-1} + 4a_{n-2} + 4a_{n-3}$

$$a_0 = 8, a_1 = 6, a_2 = 26$$

Non-homogeneous recurrence relation

Ex $a_{n+2} - 5a_{n+1} + 6a_n = 0$

with $a_0 = 1, a_1 = -1$

→ The homogeneous recurrence relation:

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

Let $a_n = r^n$

$$r^{n+2} - 5r^{n+1} + 6r^n = 0$$

$$\Rightarrow r^2 - 5r + 6 = 0$$

$$r = 2, 3$$

$$a_n^{(h)} = C_0 2^n + C_1 3^n \quad \text{--- ①}$$

Non homogeneous part sol:

$$a_n = A, a_{n+1} = A, a_{n+2} = A$$

Substituting in recurrence relation

$$A - 5A + 6A = 2$$

$$A = 1$$

$$a_n^{(p)} = 1$$

$$a_n = a_n^{(h)} + a_n^{(p)} = C_0 2^n + C_1 3^n + 1$$

$$a_0 = 1, a_1 = -1$$

$$a_0 = C_0 2^0 + C_1 3^0 + 1 = 1$$

$$\Rightarrow C_0 + C_1 + 1 = 1$$

$$\Rightarrow \boxed{C_0 + C_1 = 0}$$

$$a_1 = C_0 2^1 + C_1 3^1 + 1 = -1$$

$$\boxed{2C_0 + 3C_1 = -2}$$

$$C_0 = 2, C_1 = -2$$

$$a_n = 2 \cdot 2^n - 2 \cdot 3^n$$

$$\boxed{a_n = 2^{n+1} - 2 \cdot 3^n + 1}$$

(3) If $f(n) = \beta^n$ ($\alpha(n)$), where

β is not a characteristic root

$$a_n = \beta^n [A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m]$$

Ex solve $a_n - 7a_{n-1} + 12a_{n-2} = 2^n$

→ Homogeneous part,

$$a_n - 7a_{n-1} + 12a_{n-2} = 0$$

Let $a_n = r^n$

$$r^n - 7r^{n-1} + 12r^{n-2} = 0$$

$$\Rightarrow r^2 - 7r + 12 = 0$$

$$\boxed{r = 3, 4}$$

$$a_n^{(h)} = C_0 3^n + C_1 4^n$$

$$a_n = 2^n A_0$$

Substituting in recurrence relation

$$2^n A_0 - 7 \cdot 2^{n-1} A_0 + 12 \cdot 2^{n-2} A_0 = 2^n$$

Dividing 2^n throughout the eqn

$$A_0 - \frac{7}{2} A_0 + \frac{12}{4} A_0 = 1$$

$$A_0 - \frac{7}{2} A_0 + \frac{12}{4} A_0 = 1$$

$$\frac{A_0}{2} = 1 \Rightarrow A_0 = 2$$

$$a_n = 2 \cdot 2^n = 2^{n+1}$$

Hence, the solⁿ of recurrence relation,

$$a_n = a_n^{(ln)} + a_n^{(p)}$$

$$\text{Eg } a_{n+2} - a_{n+1} - 2a_n = n^2$$

$$\text{let, } a_{n+2} - a_{n+1} - 2a_n = 0$$

$$a_n = r^n$$

$$r^{n+2} - r^{n+1} - 2r^n = 0$$

$$\Rightarrow r^n(r^2 - r - 2) = 0$$

$$\Rightarrow r^2 - r - 2 = 0$$

$$\Rightarrow r = \frac{1 \pm \sqrt{1+4 \cdot 1 \cdot 2}}{2} = 2, -1$$

$$a_n = A_0 + A_1 n + A_2 n^2$$

$$a_{n+1} = A_0 + A_1(n+1) + A_2(n+1)^2$$

$$a_{n+2} = A_0 + A_1(n+2) + A_2(n+2)^2$$

substituting given relation.

$$A_0 + A_1(n+2) + A_2(n^2 + 4n + 4) - (A_0 + A_1(n+1) + A_2(n^2 + 2n + 1)) - 2(A_0 + A_1 n + A_2 n^2) = n^2$$

$$\Rightarrow (A_0 + 2A_1 + 4A_2 - A_0 - A_1 - A_2 - 2A_0) + (A_1 + 4A_2 - A_1 - 2A_2 - 2A_1)n + (A_2 - A_2 - 2A_2) =$$

Coefficient of n^2

$$-2A_2 = 1$$

$$A_2 = -\frac{1}{2}$$

Coefficient of n

$$2A_2 - 2A_1 = 0$$

$$-A_2 - 2A_1 = 0$$

$$A_1 = \frac{1}{2}$$

Coefficient of Const

$$-2A_0 + A_1 + 3A_2 = 0$$

$$\Rightarrow -2A_0 - \frac{1}{2} - \frac{3}{2} = 0$$

$$\Rightarrow A_0 = -1$$

$$a_n'' = -1 - \frac{1}{2}n - \frac{1}{2}n^2$$

$$a_n = C_0(-1)^n + C_1 2^n - \frac{1}{2}n^2 - \frac{1}{2}n - 1$$

② If $F(n) = \beta^n p(n)$, where β is a ch. root of multy t ,
 $p(n)$ is polynomial degree of n .

$$a_n = n^k \beta^n (A_0 + A_1 n + \dots + A_n n^n)$$

$$\textcircled{1} a_n - 7a_{n-1} + 12a_{n-2} = 4^n$$

$\Rightarrow$ put r^n

$$r^n - 7r^{n-1} + 12r^{n-2} = 0$$

$$r^2 - 7r + 12 = 0$$

$$r = 3, 4$$

$$a_n = C_0 3^n + C_1 4^n$$

$$a_n =$$

$$a_{n-1} =$$

$$a_{n-2} =$$

$$n 4^n \text{ to}$$

$$(a_n)$$

$$n 4^n -$$

$$\Rightarrow A_0^n$$

$$a_n =$$

Assign

$$\textcircled{1} a_n -$$

$$\textcircled{2} a_{n+2}$$

$$\textcircled{3} a_n -$$

Non-Homogeneous Part Solⁿ

$a_n = n \cdot 4^n A$ [It is a ch-root of multiply 1]

$$a_{n-1} = (n-1) 4^{n-1} A$$

$$a_{n-2} = (n-2) 4^{n-2} A$$

$$n 4^n A - 7(n-1) 4^{n-1} A + 12(n-2) 4^{n-2} A = 4^n$$

(Dividing 4^n)

$$n A - \frac{7}{4} (n-1) A + \frac{12}{4^2} (n-2) A = 1$$

$$\Rightarrow A \left(1 - \frac{7}{4} + \frac{3}{2} \right) + \left(\frac{3}{2} - \frac{7}{4} \right) A = 1$$

$$\frac{1}{4} A = 1$$

$$\Rightarrow A = 4$$

$$a_n(r) = n \cdot 4^n \cdot 4$$

$$= 4 \cdot 4^{n+1}$$

$$\boxed{a_n = C_1 4^n + C_2 5^n + n 4^{n+1}}$$

Assignment

$$\textcircled{1} a_n - 7a_{n-1} + 12a_{n-2} = n 4^n$$

$$\textcircled{2} a_{n+2} + 2a_{n+1} - 15a_n = 6n + 10, a_0 = 1, a_1 = -\frac{1}{2}$$

$$\textcircled{3} a_n - 7a_{n-1} + 12a_{n-2} = n 2^n$$

Generating Function

Suppose $(a_n) = \{a_0, a_1, a_2, \dots, a_n\}$

$$G(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n = \sum_{n=0}^{\infty} a_nx^n$$

Ex $(1, 1, 1, \dots, 1), a_n = (1)^n$

$$G(x) = a_0 + a_1x + \dots + a_nx^n + \dots$$

$$= 1 + x + x^2 + \dots + x^n + \dots$$

$$= \frac{1}{1-x}, |x| < 1$$

$(2, 2, \dots, 2)$

$$G(x) = 2(1 + x + x^2 + \dots + x^n + \dots)$$

$$= \frac{2}{1-x}, |x| < 1$$

$1, -1, 1, -1, \dots$

$$a_n = (-1)^n$$

$$G(x) = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{1+x}, |x| < 1$$

Ex $1, 2, 3, 4, 5, \dots$

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$xG(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$G(x) - xG(x) = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{1}{1-x}, |x| < 1$$

$$\Rightarrow (1-x)G(x) = \frac{1}{1-x}$$

$$\Rightarrow G(x) = \frac{1}{(1-x)^2}, |x| < 1$$

$$\sum_{n=0}^{\infty} a_n x^n$$

$$x \equiv 0, 1, 2, 3, \dots$$

$$\begin{aligned} \Rightarrow G(x) &= 0 + x + 2x^2 + 3x^3 + 4x^4 + \dots \\ &= x(1 + 2x + 3x^2 + 4x^3 + \dots) \\ &= \frac{x}{(1-x)^2}, |x| < 1 \end{aligned}$$

Sequence

Generating function

$$1$$

$$\frac{1}{1-x}$$

$$(-1)^n$$

$$\frac{1}{1+x}$$

$$(n+1)$$

$$\frac{1}{(1-x)^2}$$

$$\frac{1}{n!}$$

$$e^x$$

$$n$$

$$\frac{x}{(1-x)^2}$$

$$n^2$$

$$\frac{x(1+x)}{(1-x)^3}$$

$$n^3$$

$$\frac{x(x^2 - 4x + 1)}{(1-x)^4}$$

$$n^2 + n$$

$$\frac{2x}{(1-x)^3}$$

$$a^n$$

$$\frac{1}{1-ax}$$

$$na^2$$

$$\frac{a^2}{1-ax}$$

$$\sin \frac{n\pi}{2}$$

$$\frac{x}{1+x^2}$$

$$\cos \frac{n\pi}{2}$$

$$\frac{1}{1+x^2}$$

$$n \sin \frac{n\pi}{2}$$

$$\frac{1-x^2}{(1+x^2)^2}$$

Solution of recurrence relation using Generating function

Ex $a_n - 2a_{n-1} - 3a_{n-2} = 0$, $a_0 = 3, a_1 = 1$

Sol Multiply x^n throughout the (G.f) eqⁿ and taking over $n=2$ to ∞ .

$$a_n x^n - 2a_{n-1} x^n - 3a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 2 \sum_{n=2}^{\infty} a_{n-1} x^n - 3 \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad \text{--- (1)}$$

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \dots$$

$$= G(x) - a_0 - a_1 x$$

$$= G(x) - 3 - x \quad \text{--- (1)}$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots$$

$$= x (a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$= x \{G(x)\} - a_0 x \quad \left\{ \begin{array}{l} \text{Adding } a_0 x \text{ to} \\ \text{make it a G.f} \end{array} \right.$$

$$= xG(x) - 3x \quad \text{--- (2)}$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots$$

$$= x^2 (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x^2 G(x) \quad \text{--- (3)}$$

Then from (1),

$$G(x) - 3 - x - 2x(G(x) - 3) - 3x^2 G(x) = 0$$

$$\Rightarrow (1 - 2x - 3x^2) G(x) = 3 - 5x$$

$$\Rightarrow G(x) = \frac{3 - 5x}{1 - 2x - 3x^2} = \frac{3 - 5x}{-(x+1)(3x-1)}$$

(By partial fraction) $G(x) = \frac{2}{x+1} - \frac{1}{3x-1} = 2(-1)^n + 3^n$ (Ans)

For $a_n = a_{n-1} + 2n - 1, a_0 = 0$

Solⁿ Multiply x^n

$$a_n x^n = a_{n-1} x^n + 2n x^n + (-1) x^n$$

Taking sum from 1 to ∞

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} n x^n + \sum_{n=1}^{\infty} (-1) x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \dots = G(x) - a_0 = G(x) \quad (1)$$

$$\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \dots = x G(x) \quad (2)$$

$$\begin{aligned} \sum_{n=1}^{\infty} n x^n &= x + 2x^2 + 3x^3 + \dots \\ &= x(1 + 2x + 3x^2 + \dots) \\ &= \frac{x}{(1-x)^2} \quad (3) \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} (-1) x^n &= -x - x^2 - x^3 - \dots \\ &= -x(1 + x + x^2 + \dots) \\ &= \frac{-x}{1-x} \quad (4) \end{aligned}$$

Combining all the sum, (1), (2), (3), (4), we get,

$$G(x) = x G(x) + 2 \left(\frac{x}{(1-x)^2} \right) + \frac{-x}{1-x}$$

$$G(x) - x G(x) = \frac{2x}{(1-x)^2} - \frac{x}{1-x}$$

$$(1-x) G(x) = \frac{2x - x - x^2}{(1-x)^3}$$

$$G(x) = \frac{x + x^2}{(1-x)^3} = x^2$$

Assignment

Solve using Generating function

① $a_n = 3a_{n-1} + 2, a_0 = 1$

② $a_{n+2} - 2a_{n+1} + a_n = 2^n, a_0 = 2, a_1 = 1$

③ $a_n = 5a_{n-1} - 6a_{n-2}, a_0 = 1, a_1 = 4$

Graph Theory

Defⁿ: A graph $G = (V, E)$ consists of V as non-empty set of vertices (nodes) and E as a set of edges.

→ point (.) is a graph → it has one vertex.

A graph with finite number of vertices → finite graph.

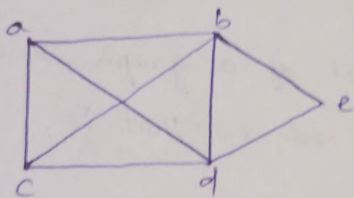
Infinite no. of vertices → Infinite graph.

→ Edges: The line connecting between 2 vertices is known as edge and is denoted as $e = \{u, v\}$, (u, v) or uv .

Degree of vertices:

The degree of vertex v in a graph G is denoted by $\deg v$ or $d(v)$ is the number of edges incident on v .

(where loop is counted twice)



$$\begin{aligned} d(a) &= 3 \\ d(b) &= 4 \\ d(c) &= 3 \\ d(d) &= 4 \\ d(e) &= 2 \end{aligned}$$

- order

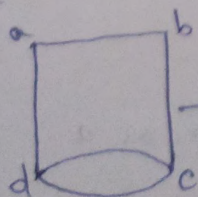
No. of vertices

- Size

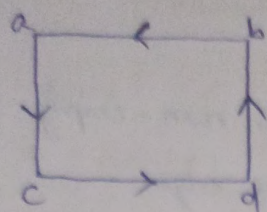
No. of edges in a graph.

Simple Graph:

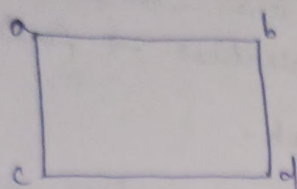
A graph in which each edge connects to two different vertices and no two edges connect the same pair of vertices (No self loop and parallel edges)



→ Not a simple graph.

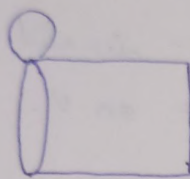


(Directed graph)



(undirected graph)

- Pseudograph = A graph with self loop



Handshaking Theorem:

The sum of the degree of vertices of a graph is equal to the twice of the number of edges. That is,

$$\sum_{i=1}^n \deg(v_i) = 2e.$$

(Applicable for simple non-directed graph)

Pf: Let G be a graph with m number of edges and n -number of vertices.

$$|V| = n, |E| = m$$

Let $V = \{v_1, v_2, \dots, v_n\}$ be the set of vertices.

We have to find degree of v_i 's for $i = 1, 2, \dots, n$

$d(v_i)$ = No. of edges incidence on v_i .

Let e be an edge, then e is either a loop or a incidence with two vertices.

→ If e is a loop, then it contributes degree of vertex 2.
 If e is incident with two distinct vertices, it also contributes 2 degree.

Similarly, for m no. of edges,

$$\text{Sum of degree} = 2 + 2 + 2 + \dots + 2 \text{ (m times)}$$

$$= 2m$$

$$= 2 \times \text{No. of edges}$$

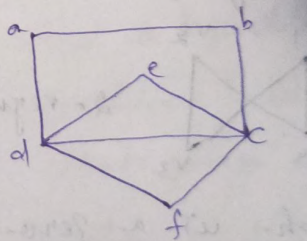
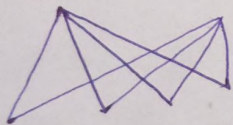
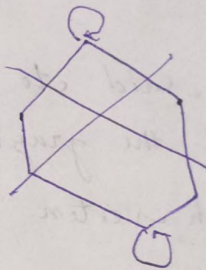
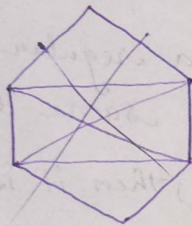
$$\sum_{i=1}^n \deg(V_i) = 2 \times \text{No. of edges}$$

$$= 2m$$

Example

Determine the no. of edges in a graph with 6 vertices
 2 of degree 4 and 4 of degree 2. (draw two such graphs)

Solⁿ



$$\sum \deg(V_i)$$

$$= 2 \times 4 + 4 \times 2$$

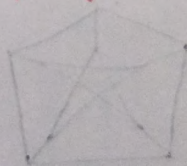
$$= 8 + 8 = 16$$

$$= 2 \times 8$$

$$= 2 \times (\text{No. of edges})$$

Assignment

Is it possible to draw a simple graph with 4 vertices and 7 edges? Justify?



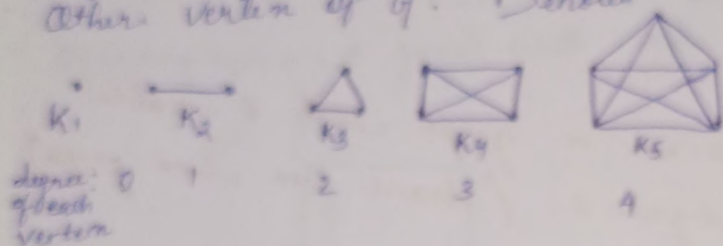
Null Graph

A graph which contains isolated vertices is known as null graph.

Eg $\therefore \dots$

Complete Graph

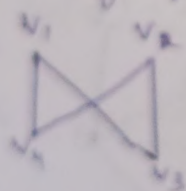
A simple graph G is said to be complete graph if each vertex of G is connected with other vertex of G . Denoted as K_n .



Regular Graph

A graph ' G ' is said to be a regular graph if all the vertices of the graph are same degree.

- If degree of each vertex is n , then it is called a n -regular graph.



2-regular graph.

Null graph is a zero regular graph.

Petersen Graph

It is a 3-regular graph.



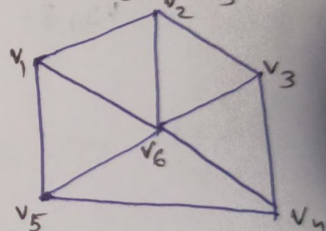
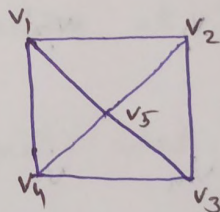
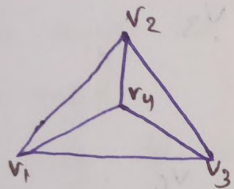
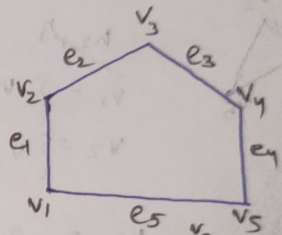
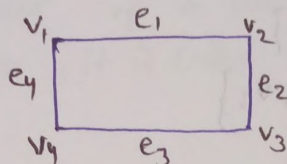
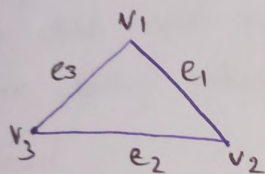
is known

Cyclic Graph

A cyclic graph ($n \geq 3$) consists of n -vertices $v_1, v_2, v_3, \dots, v_n$ and the edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

Wheel Graph

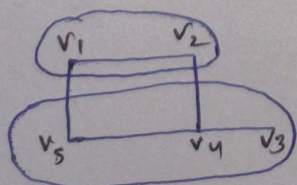
A wheel graph can be obtained by adding a vertex to a cycle C_n for $n \geq 3$ and connect the new vertex v_1 to that of the vertices of G .



Bipartite Graph

A simple graph $G(V, E)$ is called a bipartite graph if its vertex set V can be partitioned into 2 distinct subset V_1 and V_2 such that every edge in G connects a vertex in V_1 and a vertex in V_2 . No edge in G connects either 2 vertices in V_1 or two vertices in V_2 .

$$G(V, E), V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset$$

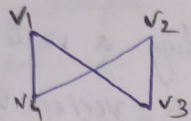


$$V_1 = \{v_1, v_4\}$$

$$V_2 = \{v_2, v_3, v_5\}$$

Complete Bipartite Graph

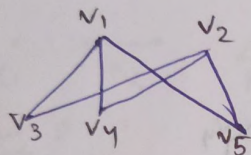
A bipartite graph G is said to be complete bipartite graph denoted by (K_m, n) , where $|V_1| = m$ and $|V_2| = n$ such that there is an edge between each pair of vertices v_1 and v_2 such that $v_1 \in V_1$ and $v_2 \in V_2$.



$$V_1 = \{v_1, v_2\}$$

$$V_2 = \{v_3, v_4\}$$

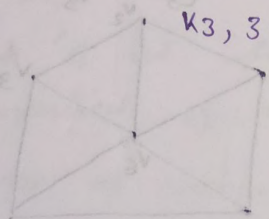
$K_2, 2$



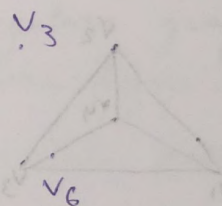
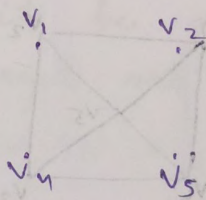
$K_2, 3$

$$V_1 = \{v_1, v_2\}$$

$$V_2 = \{v_3, v_4, v_5\}$$

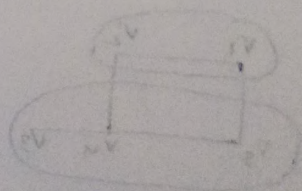


$K_3, 3 = ?$



Assignment

- 1) If G is a regular graph, find the relation b/w no. of edges and no. of vertices.
- 2) Does there exist a 4-regular graph of order 6. If such type of graph exist, then draw at least 1 graph. If not exist, then justify.



Walls :-

Let $G(V, E)$ be a graph, then a walls in G is a finite alternative of vertices and edges denoted as $v_0 e_1 v_1 e_2 v_2 e_3 v_3 \dots v_{n-1} e_n v_n$

such that, v_{k-1} and v_k are end vertices of e_k , $1 \leq k \leq n$.

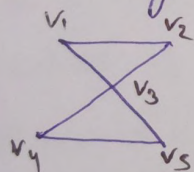
v_0 and v_n are terminal vertices and rest are called the internal vertices. All the edges and vertices may be repeated.

- If the end vertices of a walls are distinct, then it is called open walls otherwise, it is called close walls.

Trail :-

An open walls in which all the edges are distinct is called a trail.

- vertices may be repeated but edges are distinct.

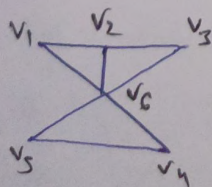


$\{v_1, v_2, v_3, v_4, v_5\}$

$\{v_1, v_3, v_4, v_5, v_3, v_2\}$

Circuit :-

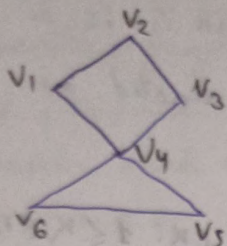
It is a closed trail in which vertices may be repeated but the edges are distinct.



$\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$

Path :-

It is a trail neither vertices nor edges are repeated.



$\{V_5, V_6, V_4, V_3, V_2, V_1\}$

Cycle:- moving on a graph such that we don't repeat the vertices as well as edges but the starting and ending vertex is same is called a cycle.

- 1> A circuit is a trail.
 - 2> A cycle is a path
 - 3> If a trail is closed, then it is a circuit.
 - 4> If a path is closed, then it is a cycle.
 - 5> Trail, circuit, path, cycle - all are walks.

Tree :-

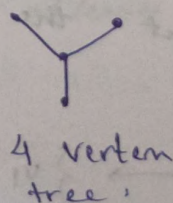
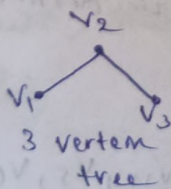
A Graph is acyclic, i.e., has no cycle.

A tree is a connected acyclic graph.

Ex: Tree with 2, 3, 4 vertices

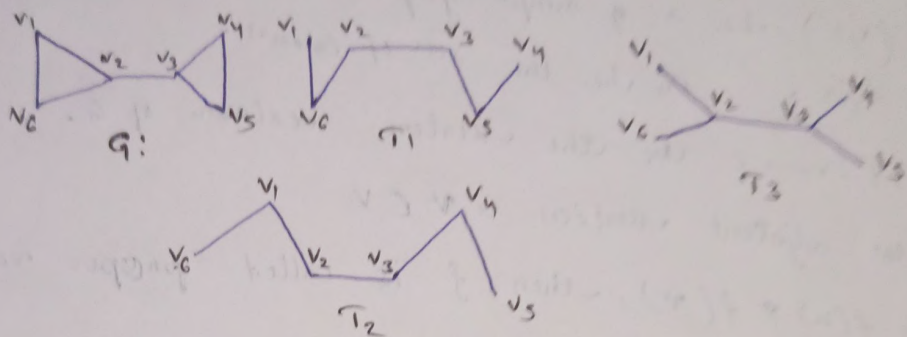
1 vertex
tree

2 vertex
tree



Spanning Tree :-

A tree is said to be a spanning tree if a connected graph G in it, it is a subgraph of G and it contains all the vertices of G .



Minimally Connected Graph :-

A connected graph G is said to be minimally connected if deletion of any edge from G disconnects the graph G .

* A Graph G is a tree if and only if G is minimally connected.

* In a tree with 2 or more vertices, there are at least 2 vertices of degree 1.

Ex Show that a k -regular graph of n -vertices exists iff kn is even and $k \leq n-1$.

Sol By defⁿ, the degree of each vertex is k and the sum of the degrees of all vertices is kn .

By handshaking theorem, the sum of degrees is equal to twice the number of edges and therefore, $kn = 2e$ (no. of edges)

The max. no. of edges is $\frac{n(n-1)}{2}$

$$kn \leq n(n-1)$$

$$\Rightarrow k \leq n-1$$

$\left. \begin{array}{l} k\text{-regular} \\ \text{graph with} \\ \text{order } n \end{array} \right\}$

Graph colouring :-

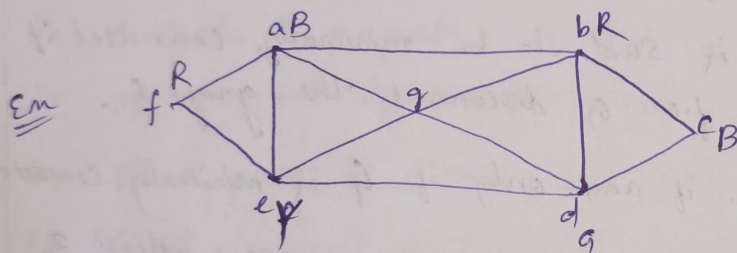
A colouring of a simple graph is the assignment of colour to each vertex of that graph so that no two adjacent vertices are assigned with same colour.

or, $G = (V, E)$ be a simple graph and we have colours $C_1, C_2, \dots, C_n$ be the set of colours.

Let, $f: V \rightarrow C$ be the vertex colour of G .

If the adjacent vertices $u, v \in V$

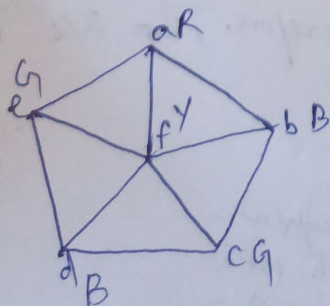
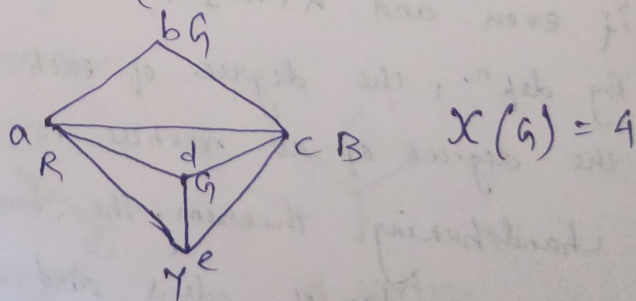
Then $f(u) \neq f(v)$, then f is called proper vertex colouring.



Chromatic Number :-

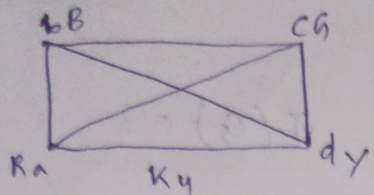
The minimum number of colours required for proper vertex colouring is known as Chromatic Number.

Denoted as $\chi(G)$,

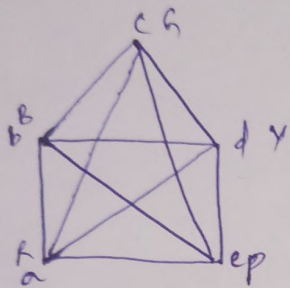


$\chi(G) = 4$

complete Graph



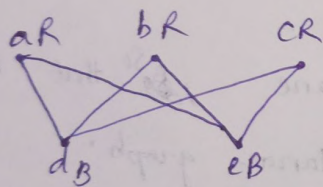
$$\chi(K_4) = 4 \text{ (No. of vertex on } K_4)$$



$$\chi(K_5) = 5$$

Chromatic Number = No of vertex.

complete Bipartite Graph

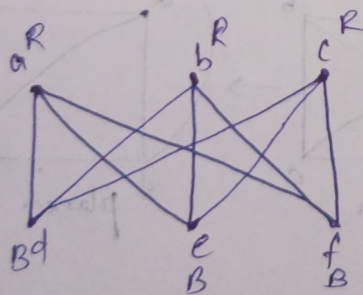


$K_{3,2}$

$$\chi(K_{3,2}) = 2$$

$$V_1 = \{a, b, c\}$$

$$V_2 = \{d, e\}$$



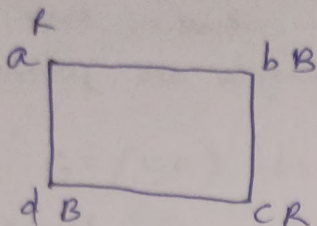
$$V_1 = \{a, b, c\}$$

$$V_2 = \{d, e, f\}$$

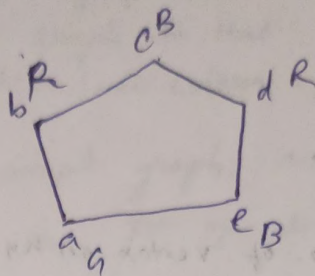
$$\chi(K_{3,3}) = 2$$

$$\boxed{\chi(K_{m,n}) = 2}$$

Cyclic Graph



$$\chi(C_4) = 2$$



$$\chi(C_5) = 3$$

$$\chi(C_n) = \begin{cases} 3 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$$

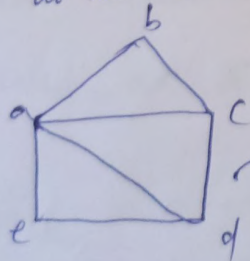
Example :-

prove that:

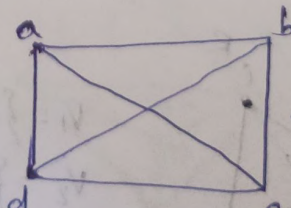
when G be a non-trivial simple graph $\chi(G) = 2$ iff G is bipartite graph.

Planar Graph :-

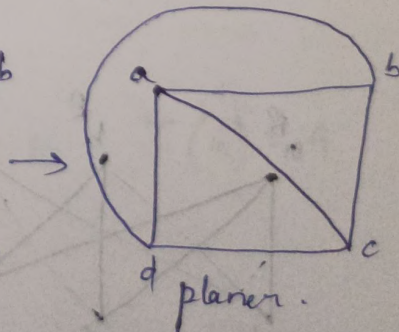
A graph which can be drawn in a plane so that its edges do not cross is said to be a planar graph.



→ planar Graph



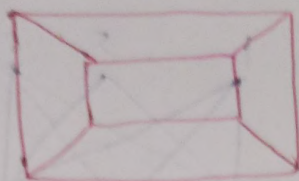
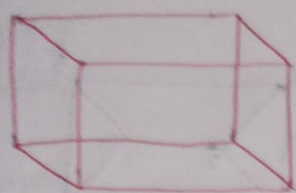
Not planar



planar.

$K_4 \rightarrow$ planar.

check K_5 is not a planar graph?



→ planar.



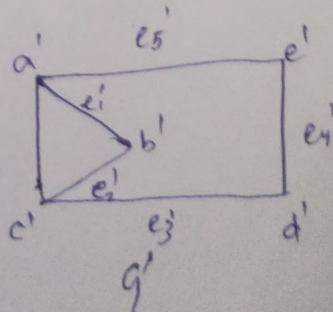
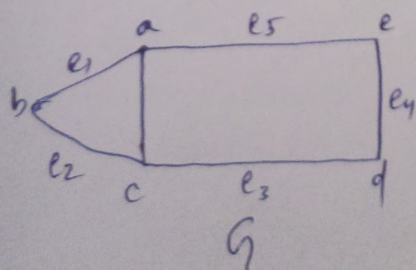
→ Not a planar graph.

Isomorphism in Graphs :-

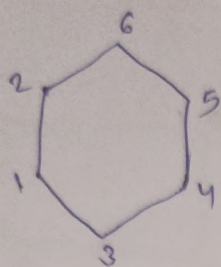
Two Graph $G(V, E)$ and $G' = (V', E')$ are said to be isomorphic if there is a bijection $f: V \rightarrow V'$ such that if u, v are adjacent in V , $f(u)$ and $f(v)$ are adjacent in V' and vice-versa.

- In other words, isomorphism is a bijection which preserves adjacency.

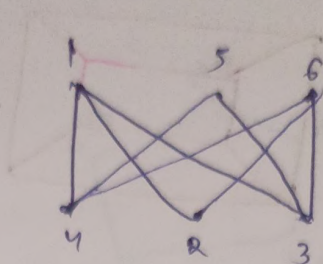
- Therefore, G_1 and G_2 are isomorphic if their vertices can be ordered in such a way that their adjacency matrices are identical.



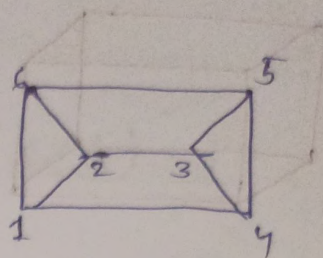
G & G' are isomorphic



G_1

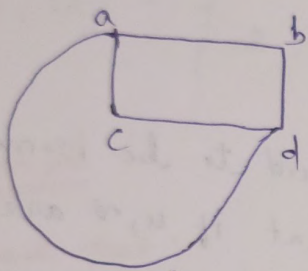


G_2

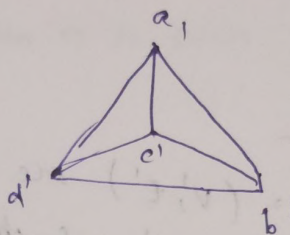


G_3

None of them are isomorphic.



G_1



G_2

Group Theory

Dt: 4/4/24

A group is a non-empty set G together with a binary operation $(\cdot, +)$ that satisfy the following properties:

- 1) Closure: for $a, b \in G$, $a \cdot b \in G$
- 2) Associative: for all $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- 3) Identity: $\exists e \in G$, such that
$$a \cdot e = e \cdot a = a$$
- 4) Inverse: $\exists a^{-1} \in G$ such that
$$a \cdot a^{-1} = a^{-1} \cdot a = e \text{ (identity)}$$

When for $a, b \in G$

$$a \cdot b = b \cdot a$$

Then G is a abelian group.

Ex integers with addition

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$$

- Real numbers with addition - form a group
- Non-Zero Real numbers with multiplication.

Ex In a group $(G, \cdot)$, the identity element is unique.

Pf - let e_1 and $e_2 \in G$ are two identity element in G .

i.e., for $a \in G$

$$a \cdot e_1 = e_1 \cdot a = a$$

$$a \cdot e_2 = e_2 \cdot a = a$$

Then,

$$a \cdot e_1 = a \cdot e_2$$

$$\Rightarrow e_1 = e_2 \text{ (by left cancellation)}$$

$\Rightarrow$ Identity

Assignment

MONOID

Identity, Closure, Associative {Satisfied}
- Inverse element is absent

Semi Group

Closure and Associative {Satisfied}
Inverse and Identity (Not Satisfied)

QUASI Group

Closure, Associative & Inverse exist here.

A monoid $(G, *)$ in which every element is inverse is called a group.

Ex Let G be a group, if $a, b \in G$ then

$$(a b)^{-1} = b^{-1} a^{-1}$$

Proof: Consider

$$\begin{aligned} & (a b) (b^{-1} a^{-1}) \\ &= a b b^{-1} a^{-1} \quad (b b^{-1} = e) \\ &= a e a^{-1} \quad (a \cdot e = a) \\ &= a a^{-1} \\ &= e \end{aligned}$$

$$\begin{aligned} \Rightarrow (a b)^{-1} &= b^{-1} a^{-1} \\ \text{or } b^{-1} a^{-1} &= (a b)^{-1} e \\ &= (a b)^{-1} \end{aligned}$$

→ The set of rational numbers $\mathbb{Q}$ is an additive abelian group.

→ The set of rationals $\mathbb{Q}$ excluding '0' is an abelian multiplicative group

$$\text{i.e. } a \in \mathbb{Q}, \frac{1}{a} \in \mathbb{Q}$$

Ques
→ The set $\{-1, 1\}$ is a multiplicative abelian group.

→ If ω is the cube root of unity, then the set $\{1, \omega, \omega^2\}$ is a multiplicative group.

→ The set M of $m \times m$ matrices is an additive abelian group.

inverse $\mathbb{Q}$

additive