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Calibration of risk aversion to real asset allocations in personal finances

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Abstract

In this thesis, we study the concept of optimal investment in a stochastic control theoretic framework to estimate an investor's risk aversion. For an investor with a life-cycle pension product and constant relative risk aversion in a Black-Scholes market, we assume their real asset allocation to be optimal. We estimate the risk aversion for optimal investments in a multidimensional market using the normal equations to solve an overdetermined system of equations. Our primary focus is estimating the risk aversion in a one-dimensional market where the investor's real asset allocation is defined as the only risky asset. We interpret the real asset allocation as a risky fund formulated in Merton's mutual funds theorem. For short-sale constrained optimal investment strategies, we propose projection strategies to estimate the risk aversion later when the constraint is no longer active. The theory is extended to cover real estate investments and a pension and real estate investment portfolio. Lastly, we suggest investing in either pension or real estate to maximize the certainty equivalent balance upon retirement, which, along with the rest of the thesis, is substantiated by a numerical study.

Contents

1 Introduction

A minority of people seek proper financial advice despite having their wealth invested in the bond market, stock market, or alternative investments. The wealth distribution between these investments can be studied in a financial market model where we can quantify an individual's risk aversion. This thesis aims to investigate new ways to estimate risk aversion and use it to provide easy access to personal, theoretical, and data-driven advice to all kinds of investors.

A way of personalizing investments is knowing the utility function of an investor. A stochastic control problem can be formulated for a given utility function to maximize future utility over investment and consumption strategies. The optimal investment strategy is closely related to the choice of utility function. Traditionally the investor is assumed to be risk-averse with diminishing marginal utility. In other words, the investor always gains more utility from obtaining more wealth, but the next unit of wealth provides less utility than the previous. This is obtained by considering a utility function exhibiting Constant Relative Risk Aversion (CRRA).

The CRRA assumption grants closed-form solutions to a handful of stochastic control problems. This includes the optimal investment problem solved in [Merton, 1969] and the further abstraction when adding human capital as in [Richard, 1975]. To interpret and simplify the theory, we adopt the way of thinking about the market through mutual funds as introduced in [Merton, 1971]. The above literature leads to an optimal investment strategy that resembles the life-cycle products seen in the Danish pension business.

It rarely happens that people know their risk aversion, and especially that they know how to quantify it. Articles like [Burgaard and Steffensen, 2020] suggest calibrating the risk aversion within a sample group based on questionnaires about certainty equivalents, i.e., how much money one would have with certainty versus the insecurity of taking a bet and potentially gaining or losing money. In this thesis, we discuss two new approaches to determining an individual's risk aversion based on market parameters and assuming that the investor's real asset allocation is optimal. Both approaches isolate risk aversion in the classical formula for the optimal weight in risky assets with human capital and mortality risk. The main result of this thesis is to use the market parameters of the investor's real asset allocation to estimate their risk aversion. In this setting, we assume that the real asset allocation is the only risky fund of the market, where a risky fund is in the sense of Merton's mutual funds theorem. The assumption simplifies the market to one dimension and grants risk aversions in the same interval as the literature suggests.

We can determine the risk aversions based on this new assumption in the three financial categories: Pension, real estate, and free funds. The categories cover typical exposures to financial risk that we can investigate in a simplified market. Risk aversion can mainly be used to compare the riskiness of different products in the market, as the unit itself has no particular meaning. The financial advice emerges from intentional or unintentional irregularities in the three risk aversions. In this thesis, we assume the free funds category to be a buffer that can grant more or less total risk. Based on all three categories, a mutual risk aversion can be estimated to grant perspective on the total risk. Initially, the advice lies in making the risk allocation deliberate and for the individual to attain the best estimate of their mutual risk aversion. To estimate the mutual risk aversion, we study an investor's total wealth. Hereafter, we seek insight into how to distribute income between categories. Obtaining more money can be used to increase risk or run off risk faster, and we will determine which strategy maximizes the certainty equivalent balance (CEB) upon retirement. The CEB is the amount of money the investor would be indifferent between receiving with guarantee instead of using the risky investment strategy. As certainty equivalents translate the risk aversion to a monetary scale, we obtain a way of communicating our results to investors in understandable terms.

This thesis is organized as follows:

Section 2 provides an introduction to the framework of stochastic control theory. We introduce the primary tool, namely the Hamilton-Jacobi-Bellman equation and the power utility function.

Section 3 introduces the financial market and the stochastic control problem that creates a life-cycle pension product. Initially, the problem is presented without a risk-free asset, but we need to add a risk-free asset to the market for a solution when adding a risk-free premium. For the rest of the thesis, we use Merton's mutual funds theorem to present the results in a market with one risky fund instead of a multidimensional market. Lastly, we add mortality risk to complete the picture.

Section 4 uses the result from the pension products to isolate the risk aversion in the equation for the optimal weight in the risky asset. Due to infeasibility in a multidimensional market, we introduce the one-fund approach and consider the market with the real pension allocation as the only risky fund instead. This is translated to the real estate category and the setup where pension and real estate are considered simultaneously.

Section 5 provides numerical results for the approaches discussed in the previous section based on real pension data. Through investment strategies, we study the relationship between the risk aversion of different life-cycle products and fabricated real estate investments. We consider the financial advice emerging from different investment strategies.

Section 6 summarizes our results from the prior sections and sheds light upon possible future studies.

Appendix A consists of additional theory on time-inconsistent problems, elaborations of the proofs in this thesis, and the parameters used from *Rådet for Afkastinvesteringer*.

Finally, all references are found at the end of this thesis.

2 Stochastic control theory

An investor wants to invest her wealth optimally in the financial market. Some of her capital gains go to consumption, and the terminal wealth is left behind as a bequest for her heirs. This section aims to formalize and solve the general problem of maximizing the utility of consumption and terminal wealth at once. The general problem is, for pension modeling purposes, split up into maximizing over the utility of terminal wealth until retirement and hereafter maximizing over the utility of consumption, i.e., the pension payments.

The first subsection formalizes the general stochastic control problem, which is solved in the second subsection. The stochastic setup follows Chapters 4 and 5 in [Bjørk, 2020], and the stochastic control theory follows Chapters 25 and 26 in [Bjørk, 2020].

The last subsection introduces the power utility function, a fundamental assumption for deriving analytical solutions to the above-formulated problem.

2.1 Formalizing the problem

The stochastic control problem is formalized in a reasonably general setup. A standard Brownian motion, W, is defined on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of W, and W represents the stochasticity in the problem. The intuition behind the theory is introduced in light of the later use in pension products in Section 3. We do so for $t \in [0, T]$ where T is some investment horizon.

Let the wealth process be controlled by consumption and a portfolio's asset allocation. In control theoretic terms, denote the control law as the set $u \in \mathbb{R}^k$, and the controlled process as the state process.

Definition 2.1. ***(State process)***
For functions

$$\alpha : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \quad \textit{and} \quad \sigma : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times d},$$

the following controlled stochastic differential equation (SDE) defines the n-dimensional state process

$$dX(t) = \alpha\left(t, X(t), \boldsymbol{u}(t, X(t))\right) dt + \sigma\left(t, X(t), \boldsymbol{u}(t, X(t))\right) dW(t). \tag{2.1}$$

We write the control law in bold, $\boldsymbol{u}$, to indicate it is a function, whereas u indicates a value in $\mathbb{R}^k$.

In this setting, the state process is partly controlled by the market and partly by the following admissible control laws.

Definition 2.2. ***(Admissible control laws)***
A control law $\boldsymbol{u}$ is called admissible if

- $\boldsymbol{u}(t,x) \in U \subset \mathbb{R}^k$ *for all* $t \in \mathbb{R}_+$ *and all* $x \in \mathbb{R}^n$.
- *For any given initial point* (t,x) *the SDE*

$$\begin{aligned} dX(s) &= \alpha\left(s, X(s), \boldsymbol{u}(s, X(s))\right) ds + \sigma\left(s, X(s), \boldsymbol{u}(s, X(s))\right) dW(s), \\ X(t) &= x, \end{aligned}$$

  has a unique solution.

The class of admissible control laws is denoted by $\mathcal{U}$.

Admissibility of the control law boils down to specific constraints on the controls imposed for practical reasons and mathematical feasibility. We will consider feedback control laws restricted to being adapted to the state process.

Definition 2.3. ***(Feedback control laws)***
Consider the feedback control law $\boldsymbol{u}(t,x) \in \mathcal{U}$, *which is the set of control laws adapted to the state process and given by*

$$\boldsymbol{u}(t, X(t)) = g(t, X(t))$$

for a deterministic function

$$g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k.$$

The construction with $g(t, X(t))$ is merely for showing adaptedness, and we continue writing $\boldsymbol{u}(t, X(t))$. Sometimes we write $\boldsymbol{u}$ instead of $\boldsymbol{u}(t, X(t))$ for notational ease and since the dependences in the control law are established in the specific control problem. In the case of dependence on a control law, $\boldsymbol{u}$ is in the superscript. As a further streamlining of notation, define the infinitesimal operator for the later use of Itô formula.

Definition 2.4. ***(Infinitesimal operator)***
Given the SDE in Equation (2.1), *the partial differential operator* $\mathcal{A}$, *referred to as the infinitesimal operator of* X, *is defined, for any function* $h(x)$ *with* $h \in C^2(\mathbb{R}^n)$ *by*

$$\mathcal{A}^{\boldsymbol{u}} h(t,x) = \sum_{i=1}^{n} \alpha_i(t,x,\boldsymbol{u}) \frac{\partial h}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} \Sigma_{ij}(t,x,\boldsymbol{u}) \frac{\partial^2 h}{\partial x_i \partial x_j}(x),$$

where

$$\Sigma(t,x,\boldsymbol{u}) = \sigma(t,x,\boldsymbol{u}) \sigma^T(t,x,\boldsymbol{u}).$$

The opportunity of gaining more wealth is not attractive for every investor if a higher risk of loss accompanies it. Thus the point is to maximize the utility of the capital gains with a utility function. In financial problems, the investor is often assumed to be risk averse with diminishing, but always positive, marginal utility. The property where an investor receives less utility for each extra unit and where the utility is increasing is captured in the concavity in Definition 2.5.

Definition 2.5. ***(Utility functions)***
A function $F : (0, \infty) \to \mathbb{R}$ where $F \in C^1$ is strictly concave and satisfies

$$F_x(0) := \lim_{x \downarrow 0} F_x(x) = \infty \quad \textit{and} \quad F_x(\infty) := \lim_{x \uparrow \infty} F_x(x) = 0,$$

is called a utility function.

Remark 2.6. *A set of functions $F(t, \cdot)$ for $t \in [0, T]$ is also called a utility function if: For every fixed $t \in [0, T]$, $F(t, \cdot)$ is a utility function in the second variable in the sense of Definition 2.5, and if $F(\cdot, x)$ is a continuous function in the first variable for every fixed positive x.*

In the following, we find a supremum of the expected utility of consumption and the expected utility of the terminal wealth. This is defined as the value function.

Definition 2.7. ***(Value function)***
Let the state process, $X(t)$, be given as in Equation (2.1) for $t \in [0, T]$ where T is some investment horizon and let $\boldsymbol{u} \in \mathcal{U}$ be an admissible control law. Define the value function

$$\mathcal{J} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$$

by

$$\mathcal{J}(t, x, \boldsymbol{u}) = E_{t,x} \left[\int_t^T F\left(s, X^{\boldsymbol{u}}(s), \boldsymbol{u}\right) ds + \Phi\left(X^{\boldsymbol{u}}(T)\right) \right],$$

where $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is a utility function for the consumption and $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the utility of bequest. The optimal value function

$$V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R},$$

is defined by

$$V(t, x) = \sup_{\boldsymbol{u} \in \mathcal{U}} \mathcal{J}(t, x, \boldsymbol{u}).$$

Here we use the notation $E[\cdot | X(t) = x] = E_{t,x}[\cdot]$ for the conditional expectation operator.

The investor aims to maximize the value function over all admissible control laws. The weapon to solve the optimization problem is a cornerstone of stochastic control theory, namely the Hamilton-Jacobi-Bellman (HJB) equation.

2.2 Solving the optimization problem

The problem of maximizing the value function of Definition 2.7 belongs to the class of time-consistent problems implying that the optimal control law is independent of the initial starting point. This is formulated in the following theorem.

Theorem 2.8. ***(Bellman's Principle of Optimality)***
Consider a fixed initial point (t, x) and consider the corresponding optimal law $\tilde{\boldsymbol{u}}_{t,x}$. Then the law $\tilde{\boldsymbol{u}}_{t,x}$ is also optimal for any subinterval of the form $[r, T]$ where $r \geq t$ and $X(r) = z$. In other words

$$\tilde{\boldsymbol{u}}_{t,x}(s, y) = \tilde{\boldsymbol{u}}_{r,z}(s, y)$$

for all $s \geq r$ and all $y \in \mathbb{R}^n$. In particular, the optimal law for the initial point $t = 0$ is optimal for all subintervals. This law will be denoted by $\tilde{\boldsymbol{u}}$.

Proof. See Theorem 25.5 in [Bjørk, 2020]. □

If $\tilde{\boldsymbol{u}}$ is the optimal control law, and the corresponding value function is the optimal value function, then they solve the supremum and the partial differential equation (PDE) in the so-called HJB equation.

Theorem 2.9. ***(The Hamilton-Jacobi-Bellman equation)***
Assume that the optimal value function V is in $C^{1,2}$ and that there exists an optimal law $\tilde{\boldsymbol{u}}$. Then the following hold:

1. *V satisfies the HJB equation*

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,x) + \sup_{u \in \mathcal{U}} \left\{ F(t,x,u) + \mathcal{A}^u V(t,x) \right\} = 0, \quad && \forall (t,x) \in (0,T) \times \mathbb{R}^n, \\ &V(T,x) = \Phi(x), && \forall x \in \mathbb{R}^n. \end{aligned}$$

2. *For each $(t,x) \in [0,T] \times \mathbb{R}^n$ the supremum in the HJB equation above is attained by $u = \tilde{\boldsymbol{u}}(t,x)$.*

Proof. A sketch of proof is given in Section 25.5 in [Bjørk, 2020]. □

A verification theorem can be formulated such that if the solution to the HJB equation is sufficiently regular, it coincides with the value function. Furthermore, the optimal control coincides with solving the supremum in the HJB equation. In other words, if V and $\tilde{\boldsymbol{u}}$ solve the HJB equation, then they solve the optimization problem.

Theorem 2.10. ***(Verification theorem)***
Suppose that we have two functions $H(t,x)$ and $g(t,x)$, such that:

- *H satisfies the condition*

$$\nabla_x H\left(s, X(s)^{\boldsymbol{u}}\right) \sigma^{\boldsymbol{u}}\left(s, X(s)^{\boldsymbol{u}}\right) \in \mathcal{L}^2,$$

  for all admissible control laws and solves the HJB equation

$$\begin{aligned} &\frac{\partial H}{\partial t}(t,x) + \sup_{u \in \mathcal{U}} \left\{ F(t,x,u) + \mathcal{A}^u H(t,x) \right\} = 0, \quad && \forall (t,x) \in (0,T) \times \mathbb{R}^n, \\ &H(T,x) = \Phi(x), && \forall x \in \mathbb{R}^n. \end{aligned}$$

- *The function g is an admissible control law.*
- *For each fixed (t,x), the supremum in the expression*

$$\sup_{u \in \mathcal{U}} \left\{ F(t,x,u) + \mathcal{A}^u H(t,x) \right\},$$

  is attained by the choice $u = g(t,x)$.

Then the following hold:

1. The optimal value function V to the control problem is given by

$$V(t,x) = H(t,x).$$

2. There exists an optimal control law $\tilde{\mathbf{u}}$, and in fact $\tilde{\mathbf{u}}(t,x) = g(t,x)$.

Proof. See Theorem 25.7 in [Bjørk, 2020] □

The verification theorem wraps up the framework of stochastic control theory. The strategy to solve a stochastic control problem is, therefore:

1. Solve the supremum in the HJB equation and insert it back into the HJB equation to get rid of the supremum.

2. Guess an ansatz for the optimal wealth function parametrized by a finite number of parameters.

3. Insert the ansatz in the HJB equation and identify the parameters that satisfy the PDE, which implicitly shows that the verification theorem is fulfilled.

2.3 Power utility

The HJB equation is, in many cases, hard to solve analytically. There are, however, known solutions if we assume the power utility function defined by

$$F(x) = \begin{cases} \frac{1}{1-\gamma} x^{1-\gamma}, & \gamma \in (0,\infty)\backslash\{1\}, \\ \log(x), & \gamma = 1. \end{cases} \tag{2.2}$$

By Remark 2.6, we add an impatience factor $\rho > 0$ such that another parametrization of the power utility function is given by

$$F(t,x) = \begin{cases} \frac{1}{1-\gamma} e^{-\rho t} x^{1-\gamma}, & \gamma \in (0,\infty)\backslash\{1\}, \\ e^{-\rho t}\log(x), & \gamma = 1. \end{cases} \tag{2.3}$$

The impatience factor discounts future consumption harder, i.e., giving the investor more utility from consuming now rather than later. In the case where $\gamma = 1$, $G(x) = \frac{1}{1-\gamma}(x^{1-\gamma} - 1)$ gives the logarithmic utility function, $G(x) = \log(x)$, for $\gamma \longrightarrow 1$. As G only differs from F in Equation (2.2) by a constant, F is said to give logarithmic utility for $\gamma = 1$ as well. The case of logarithmic utility is considered a special case in the next section.

For different parametrizations of the power utility function, the risk aversion can be compared with the Arrow–Pratt measure of relative risk aversion. We see that for both Equations (2.2) and (2.3), the Arrow–Pratt measure of relative risk aversion is constant

$$R(x) = xA(x) = \frac{-xF_{xx}(x)}{F_x(x)} = \gamma.$$

Theoretically, the power utility function exhibits Constant Relative Risk Aversion (CRRA). This type of risk aversion is in the family of Hyperbolic Absolute Risk Aversion (HARA). Members of the HARA family are known to give analytical solutions to the stochastic control problem and are studied in Section 6 in [Merton, 1971]. In [Merton, 1971], Theorem III shows that the optimal consumption and optimal weight in the risky asset in a Black-Scholes market are linear in wealth. We see this is true in the next section, but we only consider the power utility function in this thesis.

3 The optimal pension product

In this section, we aim to formalize the common understanding that pension-savers should de-escalate risk in their investment as they approach retirement. This is important as it creates a framework for life-cycle products in the Danish pension business, for which we want to estimate risk aversion.

The first subsection establishes the financial market model for n risky assets. This is relevant as the Danish pension product divides the pension-savers assets into ten asset classes. The asset classes and their expected return and volatility are provided by *Rådet for Afkastforventninger*, [Rådet for Afkastforventninger, 2022], and this market is used in the numerical study. We also add a risk-free asset as it turns out to be relevant in the analysis.

In the second subsection, we assume that there is no risk-free asset as it is neither represented in the ten asset classes nor in any pension products. The stochastic control problem is solved during the retirement phase but is infeasible in the accumulation phase.

In the third subsection, we lift the assumption by adding a risk-free asset. This solves the lack of feasibility when adding a risk-free premium to the wealth process. It also creates a life-cycle investment product that de-escalates risk as retirement approaches. As a final abstraction to the theory, mortality risk is added to the model.

The last subsection is devoted to visualizing the glide path in life-cycle products and linking the theory to real pension products.

3.1 The financial market model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq N}, \mathbb{P})$ be a filtered probability space, where $N > 0$ is some fixed investment horizon. Consider a classical Black-Scholes financial market with price processes modeled by a multidimensional geometric Brownian motion (GBM),

$$d\boldsymbol{S}(t) = \text{diag}(S_1(t), \dots, S_n(t))\boldsymbol{\alpha} dt + \text{diag}(S_1(t), \dots, S_n(t))\boldsymbol{\sigma} d\boldsymbol{W}(t),$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the expected returns, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T \in \mathbb{R}^{n \times n}$ is a symmetric and positive semi-definite variance matrix and $\boldsymbol{W} \in \mathbb{R}^n$ is a multidimensional correlated Brownian motion. Let the multidimensional correlated Brownian motion be given by

$$\boldsymbol{W} = \boldsymbol{\delta}\bar{\boldsymbol{W}},$$

where $\boldsymbol{W} \in \mathbb{R}^n$, and $\boldsymbol{\rho} = \boldsymbol{\delta}\boldsymbol{\delta}^T \in \mathbb{R}^{n \times n}$ is a symmetric and positive semi-definite correlation matrix.

Furthermore, assume the possibility of investing in the $n + 1$ risk-free asset modeled by the process

$$dB(t) = r(t)B(t)dt,$$

where $r \in \mathbb{R}_+$ is a deterministic short interest rate and strictly smaller than all entrances in $\boldsymbol{\alpha}$. The latter condition ensures we consider a market where we do not gear the risk-free asset.

3.2 The case with no risk-free asset

Consider the wealth process, X, to be the savings account of an investor with dynamics

$$dX(t) = \begin{cases} X(t)\boldsymbol{w}^T(t)\boldsymbol{\alpha}dt + l(t)dt + X(t)\boldsymbol{w}^T(t)\boldsymbol{\sigma}d\boldsymbol{W}(t), & t \in (0,T), \\ X(t)\boldsymbol{w}^T(t)\boldsymbol{\alpha}dt - \boldsymbol{c}dt + X(t)\boldsymbol{w}^T(t)\boldsymbol{\sigma}d\boldsymbol{W}(t), & t \in [T,N), \end{cases} \tag{3.1}$$

where we assume that the savings account is zero when initiated, $X(0) = 0$. As we in this section divide with the process X when $t \in (0,T)$, we do not include zero. After the initiation of the contract, the premium rate makes the savings account positive. From another perspective, it does not make sense to find the optimal investment strategy if there is nothing to invest. If a lump sum is added upon initiation, then time zero could have been included. At time N, we assume the insurance contract ends. The weight in the risky assets is given by the vector $\boldsymbol{w} \in \mathbb{R}^n$, where $\sum_{i=1}^n w_i(t) = 1$. Hence the first and the last term are the expected return from investing the account and the stochastic error term of this return. The deterministic strictly positive function l is a premium rate paid into the savings account, and the process $\boldsymbol{c}$ is the benefit rate paid from the savings account, which we control during retirement. Hence for the market provided in Section 3.1 without a risk-free asset, the control law is given by the set $\{\boldsymbol{w}\}$ for $t \in (0,T)$ and $\{\boldsymbol{w}, \boldsymbol{c}\}$ for $t \in [T,N)$.

We are dividing the lifetime of a pension-saver into two time periods:

- $(0,T)$: Accumulation phase.
- $[T,N)$: Retirement.

T is the time of retirement and the beginning of pension payments, and N is defined as in [Bjørk, 2020] such that

$$N = \inf\{t > 0 : X(t) = 0\} \wedge N',$$

where N' is a deterministic time horizon, where the contract is expired. We start by presenting the optimal policy for a retired individual, where $\boldsymbol{c}$ is withdrawn from the wealth process as pension benefits. The problem to be solved is, therefore, for all $t \in [T,N)$

$$V(t,x) = \sup_{\boldsymbol{w}\in\mathbb{R}^n,\, \boldsymbol{c}\in\mathbb{R}_+} E_{t,x}\left[\int_t^N F(s, X^{\boldsymbol{w},\boldsymbol{c}}(s))ds\right], \tag{3.2}$$

where $V(N,x) = 0$ to ensure that all savings are paid out during retirement, and F is a utility function.

Theorem 3.1. ***(Optimal controls during retirement)***
Let the wealth process be given by Equation (3.1) *for* $t \in [T, N)$. *Given the power utility function from Equation* (2.3), *the optimal value function in Equation* (3.2) *is given by*

$$V(t,x) = \begin{cases} \frac{1}{1-\gamma} x^{1-\gamma} g(t), & \gamma \in (0,\infty)\backslash\{1\}, \\ g(t)\log(x) + h(t), & \gamma = 1, \end{cases}$$

for g *and* h *given by*

$$\begin{aligned} g(t) &= e^{-\rho t}\left(\frac{\gamma}{(1-\gamma)C-\rho}\left[e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)} - 1\right]\right)^{\gamma}, & \gamma \in (0,\infty), \\ h(t) &= \begin{cases} 0, \\ -\int_t^N e^{-\rho s}\left(\rho s + \log(g(s)) + 1\right) + g(s)Cds, \end{cases} & \begin{matrix} \gamma \in (0,\infty)\backslash\{1\}, \\ \gamma = 1, \end{matrix} \end{aligned}$$

where C, $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are given by*

$$\begin{aligned} C &= \left(\boldsymbol{A}^T - \frac{1}{\gamma}\boldsymbol{B}^T\right)\left(\boldsymbol{\alpha} - \frac{1}{2}\boldsymbol{\Sigma}\left(\boldsymbol{A} - \frac{1}{\gamma}\boldsymbol{B}\right)\right), \\ \boldsymbol{A} &= \frac{1}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{\Sigma}^{-1}\boldsymbol{e} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{\Sigma}^{-1}\left(\frac{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{e} - \boldsymbol{\alpha}\right), \end{aligned}$$

for all pairs (t,x) *with* $t \in [T,N)$ *and* $x \in [0,\infty)$. *For all* $\gamma \in (0,\infty)$, *the corresponding process of the optimal fraction invested in the risky assets has the form*

$$\tilde{\boldsymbol{w}} = \boldsymbol{A} - \frac{1}{\gamma}\boldsymbol{B}, \tag{3.3}$$

and the optimal consumption process is given by

$$\tilde{\boldsymbol{c}}(t,x) = x\left(\frac{\gamma}{(1-\gamma)C-\rho}\left[e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)} - 1\right]\right)^{-1}.$$

Proof. The first part of the proof follows p. 355–358 in [Bjørk, 2020]. The final derivation of the functions is found in Appendix A.1. □

We turn to the accumulation phase after finding the optimal weights placed in the risky assets and the optimal consumption during retirement. We wish to assume a fixed positive risk-free premium paid to the savings account. However, we cannot progress this way since there is no explicit solution to the HJB equation unless the premium can be replicated in the market. Therefore, the result of this subsection can only come into use during retirement.

The future premium is referred to as human capital in the literature and is often thought of as having bond-like nature or, in this case, risk-free. This assumption creates the life-cycle products we study characterized by a glide path. Assuming human capital to be stock-like makes a hump-shaped investment profile, see [Benzoni et al., 2007]. This does not resemble the life-cycle products considered in this thesis; therefore, this assumption is disregarded.

3.3 The case with a risk-free asset

Let the market be given as in Section 3.1 containing a risk-free asset. For future references and to facilitate the results, we introduce another perspective on the market through a separation theorem, Merton's mutual funds theorem. The theorem is used to solve the stochastic control problem in a pension-saver's retirement and accumulation phase.

3.3.1 Merton's mutual funds theorem

By Merton's mutual funds theorem, every pension-saver is indifferent between investing in the n dimensional market or one specific risky fund. The pension-saver invests a fraction of her wealth in the risky fund, depending on her utility function. The rest is invested in a risk-free fund. The interpretation of investing in mutual funds is vital for later results in this thesis.

Theorem 3.2. ***(Merton's mutual funds theorem)***
Given n assets with prices S_i, whose changes are lognormally distributed, then:

1. *There exists a unique (up to a nonsingular transformation) pair of "mutual funds" constructed from linear combinations of the assets such that, independent of preferences (i.e., the form of the utility function), wealth distribution, or time horizon, individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets.*

2. *If P_f is the price per share of either fund, then P_f is log-normally distributed.*

Proof. See, [Merton, 1971] p. 384–386. □

The respective weight in the risky and risk-free funds is specified in the following corollary.

Corollary 3.3. ***(Merton's mutual funds theorem with a risk-free asset)***
If one of the assets is "risk-free" (say the $n+1$), then one mutual fund can be chosen to contain only the risk-free asset and the other to contain only the risky assets in the proportions:

$$\theta_i = \frac{\sum_{j=1}^{n} [\sigma_{ij}]^{-1} (\alpha_j - r)}{\sum_{k=1}^{n} \sum_{j=1}^{n} [\sigma_{kj}]^{-1} (\alpha_j - r)}, \qquad \forall i = 1, \dots, n, \tag{3.4}$$

where $\{\alpha_i, \sigma_{ij}\}$ define the physical distribution of the returns, $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$, and ρ_{ij} is the correlation coefficient between assets i and j.

Proof. See [Merton, 1971], p. 386–387. □

The proportions of each asset held by the risky fund can be written as a matrix such that

$$\boldsymbol{\theta} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - r\boldsymbol{e})}{\boldsymbol{e}^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - r\boldsymbol{e})},$$

where $\boldsymbol{e}$ is the n-dimensional vector of ones. Summing over the θ_i's in Equation 3.4, we see that $\boldsymbol{e}^T \boldsymbol{\theta} = 1$ corresponding to the fact that there is no risk-free asset in the risky fund. The expected return and volatility of the risky fund are found by

$$\alpha_\theta = \boldsymbol{\theta}^T \boldsymbol{\alpha} \quad \text{and} \quad \sigma_\theta = \sqrt{\boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta}},$$

and the return of the risk-free fund, θ_{n+1}, is the instantaneous interest rate, r.

Theorem 3.2 states that the price change of a share in the risky fund is log-normally distributed, implicating that the price process of the risky fund follows the GBM

$$dS(t) = S(t)\alpha_\theta dt + S(t)\sigma_\theta dW(t),$$

where

$$dW(t) = \frac{1}{\sigma_\theta}\boldsymbol{\theta}^T\boldsymbol{\sigma}d\boldsymbol{W}(t).$$

The above formulation defines a market of one risky asset, and it is now possible to go back and forth between the formulation in one dimension and n. Let the weight in the risky fund be defined as $\boldsymbol{\pi}$ and the weight in the risk-free fund as $\boldsymbol{\pi}_r$. From this point, the risky fund is also referred to as a risky asset. For the results of this section and later analysis, we define the market price of risk (Sharpe ratio) as $(\alpha_\theta - r)/\sigma_\theta$.

For the rest of the sections, we consider the market with one risky asset and one risk-free asset, where $\boldsymbol{\pi} + \boldsymbol{\pi}_r = 1$ at all time points. The risky asset has expected return α and volatility σ. Then the wealth process of a pension-saver is given by

$$dX(t) = \begin{cases} \boldsymbol{\pi}(\alpha - r)X(t)dt + (rX(t) + l(t))dt + \boldsymbol{\pi}\sigma X(t)dW(t), & t \in (0, T), \\ \boldsymbol{\pi}(\alpha - r)X(t)dt + (rX(t) - \boldsymbol{c})dt + \boldsymbol{\pi}\sigma X(t)dW(t), & t \in [T, N), \end{cases} \tag{3.5}$$

where $X(0) = 0$. The interpretation is the same as in Equation (3.1), but we have added a risk-free return on the savings account given by $(1 - \boldsymbol{\pi})rX$. Notice that as the expected return is assumed to be greater than the interest rate the control $\boldsymbol{\pi} \in \mathbb{R}_+$, and we do not have to write $|\boldsymbol{\pi}\sigma|$ in Equation (3.5).

3.3.2 Optimal controls during retirement

The optimal controls during retirement is a simplified version of Merton's portfolio problem solved in [Merton, 1969]. We state the result, disregarding the utility of bequest, as in Theorem 3.1, but including an impatience factor in the power utility function similar to what is done in the literature. The problem to be solved is, therefore, for all $t \in [T, N)$

$$\begin{aligned} V(t,x) &= \sup_{\{\boldsymbol{\pi},\boldsymbol{c}\}\in\mathbb{R}_+} E_{t,x}\left[\int_t^N F(s, X^{\boldsymbol{\pi},\boldsymbol{c}}(s))ds\right], \\ V(N,x) &= 0. \end{aligned} \tag{3.6}$$

Theorem 3.4. ***(Optimal controls during retirement)***
Let the wealth process of a pension-saver be given as in Equation (3.5) *for* $t \in [T, N)$*. Given the power utility function from Equation* (2.3)*, the optimal value function in Equation* (3.6) *is given by*

$$V(t,x) = \begin{cases} \frac{1}{1-\gamma}x^{1-\gamma}g(t), & \gamma \in (0,\infty)\backslash\{1\}, \\ g(t)\log(x) + h(t), & \gamma = 1, \end{cases}$$

for g and h given by

$$g(t) = e^{-\rho t}\left(\frac{\gamma}{(1-\gamma)C-\rho}\left(e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)}-1\right)\right)^{\gamma}, \qquad \gamma \in (0,\infty),$$

$$h(t) = \begin{cases} 0, & \gamma \in (0,\infty)\backslash\{1\}, \\ -\int_t^N e^{-\rho s}(\rho s + \log(g(s)) + 1) - g(s)C ds, & \gamma = 1, \end{cases}$$

where

$$C = r + \frac{1}{\gamma}\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2},$$

for all pairs (t, x) with $t \in [T, N)$ and $x \in [0, \infty)$. For $\gamma \in (0, \infty)$, the corresponding process of the optimal fraction invested in the risky asset has the form

$$\tilde{\pi} = \frac{(\alpha - r)}{\sigma^2}\frac{1}{\gamma}, \tag{3.7}$$

and the optimal consumption is given by

$$\tilde{\boldsymbol{c}}(t,x) = x\left(\frac{\gamma}{(1-\gamma)C-\rho}\left(e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)}-1\right)\right)^{-1}. \tag{3.8}$$

Proof. In Section 26.2 in [Bjørk, 2020], change the parametrization of the power utility function and solve with the Bernoulli equation as in the proof of Theorem 3.1. The proof of the special case, $\gamma = 1$, is given in [Klüppelberg and Pergamenchtchikov, 2005], where the impatience factor is easily added. □

Notice that the optimal value function and the optimal consumption in Theorem 3.4 are almost in the same form as in Theorem 3.1 except for the constant C and the impact this has on h. The optimal weight in the risky asset is referred to as the Merton constant. Therefore, the optimal weight held in the risk-free asset, $\tilde{\pi}_r$, is also constant. When estimating an investor's risk aversion, we are interested in the risk aversion, γ, from Equation (3.7). The risk aversion is also present in the optimal consumption from Equation (3.8). However, this thesis focuses on the risk aversion derived from investments in the financial market.

3.3.3 Optimal controls during the accumulation phase

For the general problem of this thesis, we are interested in the case of paying a fixed percentage of wages as a risk-free premium to the savings account during the accumulation phase. This is also known as a defined contribution plan. In this regard, some integrability has to be assumed.

Assumption 3.5. *Assume that l is deterministic and integrable,*

$$\int_0^T |l(s)|ds < \infty \quad \mathbb{P}-\ \ a.s.\ , \tag{3.9}$$

where T is the time of retirement.

The problem is formulated as the utility of terminal wealth since the investor wants the largest savings account to consume from when retiring. For $t \in (0,T)$, the problem is given by

$$\begin{aligned} &V(t,x) = \sup_{\boldsymbol{\pi}\in\mathbb{R}_+} E_{t,x}[F(T, X^{\boldsymbol{\pi}}(T))], \\ &\pi_r(t) + \pi(t) = 1. \end{aligned} \tag{3.10}$$

Theorem 3.6. ***(Optimal controls during the accumulation phase)***
Let the wealth process be given as in Equation (3.5) for $t \in (0,T)$ and with fixed $l(t) \equiv l$. Given the power utility function from Equation (2.3), the optimal value function in (3.10) is given by

$$V(t,x) = \begin{cases} \frac{1}{1-\gamma}\left(x + X_l(t)\right)^{1-\gamma} g(t), & \gamma \in (0,\infty)\backslash\{1\}, \\ g(t)\log(x + X_l(t)) + h(t), & \gamma = 1, \end{cases} \tag{3.11}$$

where

$$h(t) = \begin{cases} 0, & \gamma \in (0,\infty)\backslash\{1\}, \\ e^{-\rho T} C(T-t), & \gamma = 1, \end{cases}$$

and for $\gamma \in (0,\infty)$ then g and X_l are given by

$$\begin{aligned} g(t) &= e^{-\rho t} e^{((1-\gamma)C-\rho)(T-t)}, \\ X_l(t) &= l\left[\frac{1-e^{-r(T-t)}}{r}\right], \end{aligned}$$

where

$$C = r + \frac{1}{2}\frac{1}{\gamma}\frac{(\alpha-r)^2}{\sigma^2},$$

for all pairs (t,x) with $t \in (0,T)$ and $x \in (0,\infty)$. For $\gamma \in (0,\infty)$, the corresponding process of the optimal fraction invested in the risky asset has the form

$$\tilde{\boldsymbol{\pi}}(t,x) = \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{x + X_l(t)}{x}, \tag{3.12}$$

where nothing is invested if x is zero.

Proof. The proof follows Section 4 in [Devolder et al., 2003] closely. We use another parametrization of the power utility function, and hence the parametrized optimal value function is given by $V(t,x) = \frac{1}{1-\gamma} g(t)(x + a(t))^{1-\gamma}$. The terminal condition $g(T) = e^{-\rho T}$ reflects the inclusion of the impatience factor in the utility function and $a(T) = 0$. In the special case $\gamma = 1$, let the parametrized optimal value function instead be $V(t,x) = g(t)\log(x + a(t)) + h(t)$ with the same boundary conditions on g and a, and with $h(T) = 0$. Then the proof follows from the general case, $\gamma \in (0,\infty)\backslash\{1\}$. □

Remark 3.7. *By Theorem 3.4 in [Korn and Krekel, 2003], the result also holds for discrete and continuous deterministic payment streams on the form*

$$X_l(t) = \int_t^T e^{-r(t-s)} l(s)ds + \sum_{i=1}^m D_i e^{-r(t_i - t)},$$

given a set of discrete streams D_i at times $t_i, i = 1,\ldots,m$ with $0 \le t_1 < \ldots < t_m \le T$.

The problem presented in [Devolder et al., 2003] is to find the optimal investment strategy in the retirement phase for guaranteed benefits. Under this guarantee, the optimal investment strategy is not constant after retirement. For optimal unguaranteed benefits, the optimal investment strategy is constant during retirement as shown in Theorem 3.4. This thesis focuses on the latter pension product as the glide path of this investment strategy is comparable to those studied in the Danish pension business.

The result in Equation (3.12) is also presented in Equation (71) for general HARA utility functions in [Merton, 1971]. The formula for optimal weight in the risky asset shows that the investor will hold an increased amount early while the human capital is greater than the wealth process. Then the amount in the risky asset declines rapidly as X_l goes to zero when approaching retirement, i.e., for $t \longrightarrow T$ then $\tilde{\boldsymbol{\pi}} \longrightarrow \frac{(\alpha-r)}{\sigma^2}\frac{1}{\gamma}$, the Merton constant. Therefore, we can continue with the optimal investment strategy given in (3.12) for the whole period $(0, N)$. Recall that the fraction $(\alpha - r)/\sigma^2$ is another parametrization of the Sharpe ratio defined in Section 3.1, where we consider the variance instead of the volatility. The fraction describes the relationship between the fund's expected return and riskiness. We conclude that for a high Sharpe ratio, the investor should optimally invest more in the market, which will be a relevant observation later.

3.3.4 Including mortality risk

The following is based on the modeling of optimal investment and consumption in a defined contribution plan, including the risk of dying, see [Konicz et al., 2015]. Until now, we handled the accumulation and retirement phases separately, but in [Konicz et al., 2015], it is argued to study the two phases simultaneously by letting the optimal value function of the retirement phase be the boundary condition for the accumulation phase.

By p. 194 in [Richard, 1975], we add to Merton's mutual funds theorem that the choice of investing in the risky fund of the market is also independent of the probability distribution of lifetime and life insurance opportunities. We continue to model the weight in a risky fund. Assume the survival model where the uncertain lifetime is modeled by a finite Markov chain Z on the measurable space $(\Omega, \mathcal{F})$. Assume two equivalent probability measures $\mathbb{P}$ and $\mathbb{P}^*$ on the space $(\Omega, \mathcal{F})$. The first measure is the objective measure, and $\mathbb{P}^*$ is used for pricing market and insurance risk by the pension fund. Associated with each measure, we have mortality rates measuring the probability of dying in an infinitesimal time period, respectively $\mu(t)$ and $\mu^*(t)$. The mortality rate under the objective measure will be used to discount the utility to account for the possibility of dying. Since the pension-saver has no bequest motive, she is paid an additional income of $\mu^*(t)X(t)$ over the whole period $(0, N)$ from the pension fund to be her only inheritor. The wealth function is given by

$$
\begin{aligned}
dX(t) &= \begin{cases} \left(r + \boldsymbol{\pi}(\alpha - r) + \mu^*(t)\right) X(t)dt - \boldsymbol{c}dt + \boldsymbol{\pi}\sigma X(t)dW(t), & t \in (0, T), \\ \left(r + \boldsymbol{\pi}(\alpha - r) + \mu^*(t)\right) X(t)dt + l(t)dt + \boldsymbol{\pi}\sigma X(t)dW(t), & t \in [T, N), \end{cases} \\
X(0) &= 0,
\end{aligned}
\tag{3.13}
$$

where l is deterministic and positive premium in the accumulation phase, and in the retirement phase $\boldsymbol{c} \geq 0$ is the benefits and a part of the optimization problem. The optimization problem for $t \in (0,T)$ is given by

$$V(t,x) = \sup_{\boldsymbol{\pi}\in\mathbb{R}_+} E_{t,x}\left[F(X^{\boldsymbol{\pi}}(T))\right], \tag{3.14}$$

where $\boldsymbol{\pi}_r + \boldsymbol{\pi} = 1$. The weight constraint also applies to the optimization problem for $t \in [T,N)$

$$\begin{aligned} V(t,x) &= \sup_{\{\boldsymbol{\pi},\boldsymbol{c}\}\in\mathbb{R}_+} E_{t,x}\left[\int_t^N e^{-\int_t^s \mu(\tau)d\tau} F(s,\boldsymbol{c})ds\right] \\ V(N,x) &= 0, \end{aligned} \tag{3.15}$$

where at time N, the investor is assumed to be dead with certainty.

Theorem 3.8. ***(Optimal controls for the whole life-cycle)***
Given the optimizing problems in Equations (3.14) *and* (3.15) *with wealth process as given in Equation* (3.13) *and utility function as given in Equation* (2.3)*, then the optimal value function is given by*

$$V(t,x) = \begin{cases} \frac{1}{1-\gamma} g(t)^\gamma (x + X_l^\mu(t))^{1-\gamma}, & \gamma \in (0,\infty)\backslash\{1\}, \\ g(t)\log(x + X_l^\mu(t)) + h(t), & \gamma = 1, \end{cases}$$

where

$$h(t) = \begin{cases} 0 & \gamma \in (0,\infty)\backslash\{1\}, \\ e^{-\int_t^T \mu(\tau)d\tau} \int_t^T \left(\mu^*(s) + C\right) g(s)ds + e^{-\int_t^T \mu(\tau)d\tau} h(T), & t \in (0,T),\ \gamma = 1, \\ -\int_t^N e^{-\int_t^s \mu(\tau)d\tau} \left(e^{-\rho s}\left(\rho s + \log g(s) + 1\right) - \left(\mu^*(s) + C\right) g(s)\right) ds, & t \in [T,N),\ \gamma = 1, \end{cases} \tag{3.16}$$

and for $\gamma \in (0,\infty)$ *then* g *and* X_l^μ *are given by*

$$g(t) = \begin{cases} e^{-\frac{1}{\gamma}\int_t^T (\mu(\tau) - (1-\gamma)(\mu^*(\tau)+C)d\tau} g(T), & t \in (0,T), \\ \int_t^N e^{-\frac{1}{\gamma}\int_t^s (\mu(\tau) - (1-\gamma)(\mu^*(\tau)+C)d\tau} e^{-\frac{1}{\gamma}\rho s} ds, & t \in [T,N), \end{cases} \tag{3.17}$$

$$X_l^\mu(t) = \begin{cases} \int_t^T \mathrm{e}^{-\int_t^s (r+\mu^*(\tau))\mathrm{d}\tau} l(s)\ \mathrm{d}s, & t \in (0,T), \\ 0, & t \in [T,N), \end{cases}$$

for

$$C = r + \frac{1}{2}\frac{1}{\gamma}\frac{(\alpha - r)^2}{\sigma^2}.$$

For $\gamma \in (0,\infty)$ *the optimal controls are given by*

$$\tilde{\boldsymbol{\pi}}(t,x) = \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{x + X_l^\mu(t)}{x}, \quad t \in (0,N), \tag{3.18}$$

where nothing is invested if x *is zero and*

$$\tilde{\boldsymbol{c}}(t,x) = x e^{-\frac{1}{\gamma}\rho t} g(t)^{-1}, \quad t \in [T,N).$$

Proof. See [Konicz et al., 2015] page 613–614. The case where $\gamma = 1$ follows from the proofs of

Theorem 3.4 and 3.6. □

The optimal asset allocation during retirement is unchanged from Theorem 3.6 and given by the Merton constant. The only difference in the optimal asset allocation during the accumulation phase is the human capital which is now considered to be a life annuity describing the present value of future premiums when including the risk of dying. In Equations (3.16) and (3.17), the terms $h(T)$ and $g(T)$ stem from using the optimal value function of the retirement phase as the terminal condition in the accumulation phase. Hence if we are not interested in the expected utility over the entire life-cycle but only in the accumulation phase, then we can set $g(T) = e^{-\rho T}$ and $h(T) = 0$. Notice if we do this and let $\mu = \mu^* = 0$, we get the same optimal value function and optimal consumption as in Theorems 3.4 and 3.6.

To study risk aversion, we derive some analytical properties about the stochasticity of Equation (3.18). In Remark 3.9, we show that $\{X(t) + X_l^{\mu}(t)\}_{t\geq 0}$ is a GBM that we can study in expectation.

Remark 3.9. *Notice that by inserting $\tilde{\pi}$ from Equation (3.18) into the dynamics of the wealth process and reorganizing, then*

$$\begin{aligned} d\left(X^{\tilde{\pi}}(t) + X_l^{\mu}(t)\right) = &\left(r + \mu^*(t) + \frac{1}{\gamma}\left(\frac{\alpha - r}{\sigma}\right)^2\right)\left(X^{\tilde{\pi}}(t) + X_l^{\mu}(t)\right)dt \\ &+ \frac{1}{\gamma}\frac{\alpha - r}{\sigma}\left(X^{\tilde{\pi}}(t) + X_l^{\mu}(t)\right)dW(t), \end{aligned}$$

which constitutes a GBM such that for $\lim_{t\downarrow 0} X^{\tilde{\pi}}(t) = x_0 > 0$, then

$$X^{\tilde{\pi}}(t) + X_l^{\mu}(t) = (x_0 + X_l^{\mu}(0))\underbrace{e^{\left(r + \frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{1}{\gamma}\frac{\alpha-r}{\sigma}\right)^2\right)t + \bar{\mu}^*(t) + \frac{1}{\gamma}\frac{\alpha-r}{\sigma}W(t)}}_{=Y(t)},$$

where $\bar{\mu}^*(t) = \int_0^t \mu^*(s)ds$, and with expected value

$$E\left[X^{\tilde{\pi}}(t)\right] + X_l^{\mu}(t) = (x_0 + X_l^{\mu}(0))\, e^{\left(r + \frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t + \bar{\mu}^*(t)}.$$

For $l(t) \equiv l$, the ratio between the human capital and the wealth process is given by

$$\frac{X_l^{\mu}(t)}{X^{\tilde{\pi}}(t)} = \frac{l\int_t^T e^{-\int_t^s (r+\mu^*(\tau))d\tau}ds}{l\left(\frac{1}{l}x_0 + \int_0^T e^{-\int_0^s (r+\mu^*(\tau))d\tau}ds\right)Y(t) - l\int_t^T e^{-\int_t^s (r+\mu^*(\tau))d\tau}ds}.$$

If $x_0 = l$, then the above ratio is independent of l. In all other cases, the optimal investment is dependent on the structure of l.

Previously, we assumed no one relied on the investor, and the goal was to maximize her interests. For instance, one would be interested in hedging the human capital by buying life insurance in a household. If the investor dies, she leaves her savings account, X, behind and a death sum, I, to her heirs. The premium for the coverage is given by $\mu^*(t)I(t)dt$ and is subtracted from the wealth process. The problem can be solved for optimal investment, consumption, and sum insured as in [Richard, 1975]. The weight in the risky asset is the same as in Theorem 3.8, but it is now also possible to study the risk aversion coming from the optimal sum insured. We do not progress further this way but focus solely on the risk aversion coming from investment actions in the market.

3.4 Life-cycle pension products

Now that we have introduced the theory behind life-cycle products, we want to establish the visual identity and link them to the current Danish pension products.

In the first example, we use simulation to overcome the limitations of the theory and use the short-selling constraint to create the characteristic glide path. The second example illustrates age-based glide paths of products in the Danish pension business. The third example is based on an abstraction of the stochastic control theory we have presented, namely when the problem is time-inconsistent.

As shown in Remark 3.9, different structures of l give rise to different glide paths of the life-cycle product. To find consistent risk aversions based on the glide paths shown in Example 3.11, we must keep all other variables fixed. This ensures comparability between the risk aversion of different pension-savers. The comparability might only be needed in homogeneous groups of investors, e.g., groups with the same starting age or retirement age. Pension-savers moving their account from a with-profit pension product to a life-cycle product might have had other savings account developments making this group tricky. For all homogeneous groups and all financial categories, we make the following assumption in this thesis.

Assumption 3.10. *We assume the premium is a fixed and strictly positive real number,* $l \in \mathbb{R}_+ \backslash \{0\}$.

Example 3.11. Consider an investor at age $y = 25$, with retirement at age 65, i.e., $T = 40$, power utility function, and risk aversion $\gamma = 5$. The investor places money in a savings account, taking into account the risk of dying, where the mortality risk given by the G82 basis, [P+, 2022]

$$\mu_y^*(t) = 0.0005 + 10^{5.728+0.038(y+t)-10}.$$

The market contains one risky asset and one risk-free asset with parameters

$$\alpha = 0.055, \quad \sigma = 0.154, \quad r = 0.004.$$

The savings account accumulates according to the SDE

$$dX(t) = \boldsymbol{\pi}^{(0)}(\alpha - r)X(t)dt + ((r + \mu_y^*(t))X(t) + l)dt + \boldsymbol{\pi}^{(0)}\sigma X(t)dW(t),$$

where the constrained optimal weight in the risky asset is given by

$$\boldsymbol{\pi}^{(0)}(t, X(t)) = \min\left(1, \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{X(t) + l\int_t^T e^{-\int_t^s (r+\mu_y^*(\tau))d\tau} ds}{X(t)}\right). \tag{3.19}$$

In other words, $\boldsymbol{\pi}^{(0)}$ makes sure the pension company does not short the risk-free asset and hence gear the investment of the pension-saver. The process X is simulated as an Euler scheme with a GBM starting every year. The savings account is increased by $l = 100,000$ every year and starts at zero. The Euler scheme is simulated 10,000 times and visualized with a box and whiskers plot, illustrating the variation of the simulated sample paths.

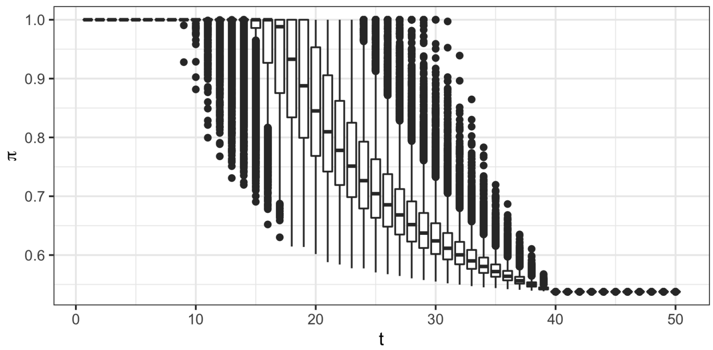

Figure 3.1: *Weight in a risky asset, y-axis, over time, x-axis, based on 10,000 simulations of the wealth process.*

The boxes in Figure 3.1 show the first and the third quantile, and the horizontal line is the median, while the dots are outliers. All simulations represent a glide path, but the market trend significantly impacts when risk de-escalation begins. The investor is, in other words, *supposed* to change her asset allocation depending on the market.

The reason for the glide path is the ratio between the savings account and the human capital, and the point where the de-escalation of risk starts stems from $x \mapsto \min(1, x)$. If $\tilde{\boldsymbol{\pi}}$ from Theorem 3.8 were considered, then a higher fraction would be invested in the market. Thus X would, in expectation, grow faster, and the decline would be steeper and happen earlier.

△

Example 3.11 leads to dividing the accumulation phase into two periods: The (sub-optimal) gearing phase and the (optimal) de-escalation phase. The gearing phase is thus from the pension product's start until the risk de-escalation begins, and the rest of the accumulation phase is the de-escalation phase. The investment constraint in the gearing phase makes the optimal value function derived in Theorem 3.6 incorrect. As the optimal value function is the expected future utility, it is again correct after the gearing phase.

The following example shows the deterministic glide paths in Danish life-cycle products.

Example 3.12. An age-dependent glide path characterizes the life-cycle pension products on the market. This is, of course, a less personal product than Example 3.11, as it does not include dependence on the savings account. The grey filling, i.e., weight in stocks in Figure 3.2 illustrates the age-dependent glide path.

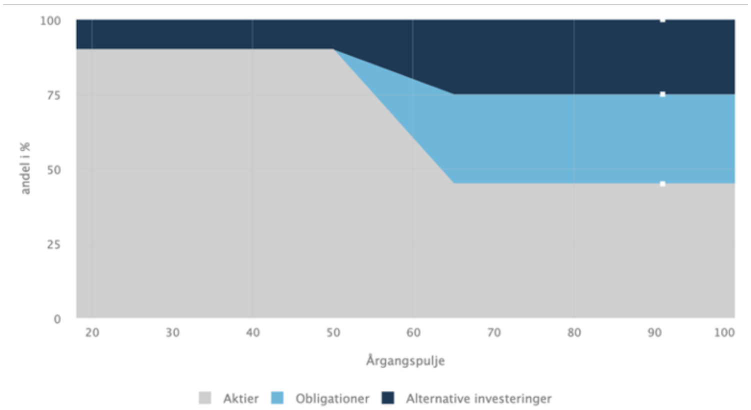

Figure 3.2: *The x-axis is age, and the y-axis is the weight invested in stocks (grey), bonds (light blue), and alternative investments (dark blue) in a high-risk life-cycle product from Sampension, [Sampension, 2022].*

Figure 3.3 shows that PFA has two risky funds, high and low risk, which combined give different risk profiles, all with an age-dependent glide path.

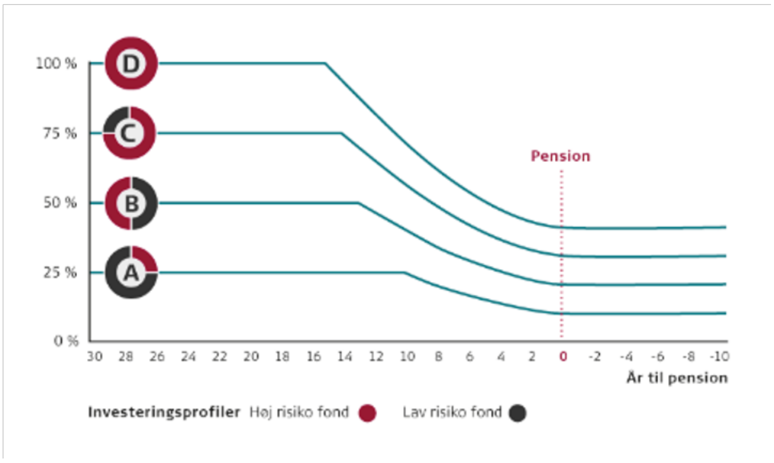

Figure 3.3: *The x-axis is the time until retirement, and the y-axis is the weight invested in a high-risk fund in the life-cycle products from PFA, [PFA, 2022].*

The above pension products offer high risk early in the life-cycle, where it is possible to recover if the pension-saver faces a great loss. It seems reasonable that the pension product decreases riskiness when approaching retirement so that the pension-saver is more certain about the savings account's size during retirement.

△

The third example introduces the concept of certainty equivalents, the amount of money it takes for the investor to be indifferent between a gamble or getting an amount of money for certain. In Example 3.13, we maximize over the expected terminal certainty equivalents and show another glide path from an insurance portfolio point of view. The example relies on a result from the stochastic control theory of time-inconsistent problems, where the necessary results can be found in Appendix A.2. Time-inconsistency covers the situation where the optimal investment strategy chosen at a specific time point is not necessarily optimal for all time points – unlike the Bellman optimality principle in Definition 2.8.

Example 3.13. Assume two investors with utility functions given by

$$u_i(x) = x^{1-\gamma_i}, \quad \text{for} \quad i \in \{1, 2\},$$

who want to maximize their sum of expected terminal certainty equivalents. These are defined as

$$u_i^{-1}\Big(\mathrm{E}_{t,x}\left[u_i\left(X^\pi(T)/2\right)\right]\Big)$$

for $i \in \{1, 2\}$ and fixed retirement at time T. Time-inconsistency arrives from taking the inverse utility function of the expected value. The calculations can be found in Appendix A.3, giving the following weight to the risky asset

$$\tilde{\boldsymbol{\pi}} = \frac{(\alpha - r)\left(b_1(t) + b_2(t)\right)}{\sigma^2\left(\gamma_1 b_1(t) + \gamma_2 b_2(t)\right)},$$

where b_1 and b_2 is a system of Ordinary Differential Equations (ODEs) given by

$$0 = -b_i'(t) - \frac{(\alpha - r)^2\left(b_1(t) + b_2(t)\right)}{\sigma^2\left(\gamma_1 b_1(t)\gamma_2 b_2(t)\right)} b_i(t) - r b_i(t) + \frac{1}{2}\frac{(\alpha - r)^2\left(b_1(t) + b_2(t)\right)^2}{\sigma^2\left(\gamma_1 b_1(t)\gamma_2 b_2(t)\right)^2}\gamma_i b_i(t),$$

where $b_i(T) = \frac{1}{2}$ for $i \in \{1, 2\}$.

We solve the system of ODEs numerically for $\gamma_1 = 0.5$, $\gamma_2 = 2.5$, $\alpha = 0.055$, $\sigma = 0.154$, and $r = 0.004$. Then we use the function $\min(1, \tilde{\boldsymbol{\pi}})$ such that the weight in the risky asset is set to one if the risk-free asset is shorted. Figure 3.4 shows a deterministic age-dependent glide path towards retirement as in Example 3.12. However, the glide path comes from multiple individuals with different risk aversions participating in a pension pool.

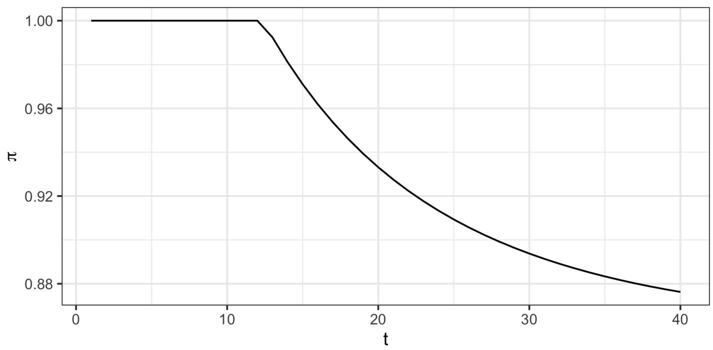

Figure 3.4: *Weight in the risky asset, y-axis, over time, x-axis, for two investors who want to maximize their expected terminal certainty equivalents.*

△

Being in a pension pool is inevitable unless we consider pure unit link products where the investor chooses her specific assets. Despite the effect in Example 3.13 being present in life-cycle pension products, we are not pursuing this theory any further.

4 Estimating risk aversions

The previous section provided a way of finding the optimal asset allocation, i.e., the optimal weight in the risky assets. In this section, we formalize the idea of assuming the pension-saver's real asset allocation as the optimal asset allocation and then estimate the risk aversion.

The first subsection considers the disadvantages of studying the optimal asset allocation in n dimensions. Our main concerns are the correlation in the market and assuming a constrained optimal pension allocation to be the optimal asset allocation. It is shown in the numerical study that the risk aversions estimated with this method are too high or even negative. We instead assume that the real asset allocation is a risky fund according to Merton's mutual funds theorem. We denote this the *one-fund approach* as we reduce the market to only consist of one risky fund making the problem one-dimensional. With the one-fund approach, we can handle the retirement and de-escalation phases. However, in the gearing phase, where the pension-saver is investing under the short-selling constraint, we cannot estimate a meaningful risk aversion. We consider different age-dependent glide paths of a pension-saver in the gearing phase to project the risk aversion to the end of the gearing phase. We elaborate on the interpretation of risk aversion over time and compare different age-dependent glide paths in the numerical study.

In the second and third subsections, we extend the setup and methods developed within the pension category to the other financial categories: Real estate and free funds.

In the fourth subsection, we consider an investor's total wealth and suggest how to find the risk aversion based on all categories simultaneously.

The last subsection describes how certainty equivalents can be used as a metric to compare risk aversions and investment strategies. We explain why they are easier to interpret and compare than the estimated risk aversions. In the numerical study, we use certainty equivalent balance upon retirement to produce theoretically founded investment advice.

4.1 Pension

When a savings account is invested, it is usually without short sales, but the theory suggested so far in this thesis is not under this constraint. The literature only covers the case of optimal investment under the short-selling constraint *without human capital*, see [Teplá, 2000]. The short-selling constraint influences the structure of the risky fund and the weight placed in the risky fund since the individual assets and the risk-free fund, respectively, cannot be shorted. For this section, we consider $t \in (0, N)$, as we have concluded that the optimal investment in the accumulation phase can determine the formula for optimal investment in the whole interval.

In the first approach, we disregard the short-selling constraint and establish the impact of assuming the asset allocation in a pension product is an optimal solution.

The second approach is the main result of this thesis. This approach does not prohibit short-selling but shifts the focus from the weight placed in the risky fund to the relationship between the real pension portfolio's expected return and volatility.

4.1.1 The first approach

Let $t \in (0, N)$ be today then by Theorem 3.8, the optimal weight in the risky fund is given by

$$\tilde{\boldsymbol{\pi}} = \frac{\alpha_\theta - r}{\sigma_\theta^2}\frac{1}{\gamma}\frac{X(t) + X_l^\mu(t)}{X(t)},$$

where α_θ and σ_θ are parameters of the risky fund $\boldsymbol{\theta}$, X_l^μ is the human capital and X is the wealth process. We have assumed that $X(0) = 0$, equivalent to opening the savings account, and after the first premium rate at time $0 + dt$, then $X(t) > 0$. Therefore we do not divide with zero in the above formula for the optimal asset allocation. Then by Merton's mutual funds theorem, we have the optimal portfolio for the pension allocation given by

$$\tilde{\boldsymbol{w}} = \boldsymbol{\theta}\tilde{\boldsymbol{\pi}}. \tag{4.1}$$

The investor is indifferent between choosing the optimal portfolio in Equation (4.1) and the optimal portfolio

$$\tilde{\boldsymbol{w}} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - r\boldsymbol{e})\frac{1}{\gamma}\frac{X(t) + X_l^\mu(t)}{X(t)}, \tag{4.2}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$ are the physical parameters of the market. This result has the same structure as Equation 3.12 and comes from a similar proof of Theorem 3.8, but with matrices.

If the asset allocation in the pension product is assumed to be optimal, then isolating $1/\gamma$ in Equation (4.2) results in n equations with one unknown, γ. This is an overdetermined system of equations that do not have a unique solution. A standard method to solve overdetermined systems is to approximate the solution by minimizing the euclidean norm, [Williams, 1990],

$$\min_\gamma \left|\left|\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - r\boldsymbol{e})\frac{1}{\gamma}\frac{X(t) + X_l^\mu(t)}{X(t)} - \tilde{\boldsymbol{w}}\right|\right|. \tag{4.3}$$

Equation (4.3) is on the form $\min_y \mid\mid \boldsymbol{A}\frac{1}{y} - \boldsymbol{b} \mid\mid$, and the y that provides the smallest norm can be formulated via the normal equations.

Proposition 4.1. ***(Normal equation)***
Given the matrix equation

$$\boldsymbol{A}y = \boldsymbol{b},$$

for $\boldsymbol{A} \in \mathbb{R}^{n\times k}$ *and* $\boldsymbol{b} \in \mathbb{R}^{n\times 1}$ *for* $n > k$. *The solution that minimizes the norm*

$$\min_y \mid\mid \boldsymbol{A}y - \boldsymbol{b} \mid\mid,$$

is given by the normal equation

$$y = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}.$$

The following example shows the complications with this approach, but for $n = 2$, where the inverse of the variance matrix is easier to interpret.

Example 4.2. ***(Risk aversion in a market with two risky assets and one risk-free)*** Consider a market with two risky assets with

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\rho \in (-1, 1)$ is the correlation between the assets. Without loss of generality, let $r = 0$ for the risk-free asset. Consider the optimal weight in the risky assets for a retired pension-saver

$$\tilde{\boldsymbol{w}} = \frac{1}{\Sigma_{11}\Sigma_{22} - \Sigma_{12}\Sigma_{21}} \begin{pmatrix} \Sigma_{22} & -\Sigma_{21} \\ -\Sigma_{12} & \Sigma_{11} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \frac{1}{\gamma}. \tag{4.4}$$

Let the first fraction be denoted by k, which is a positive constant since $|\rho| \neq 1$. Then Equation (4.4) is given by

$$\tilde{\boldsymbol{w}} = k \begin{pmatrix} \alpha_1\Sigma_{22} - \alpha_2\Sigma_{21} \\ \alpha_2\Sigma_{11} - \alpha_1\Sigma_{12} \end{pmatrix} \frac{1}{\gamma}. \tag{4.5}$$

Consider the three cases for the vector in Equation (4.5):

1. Assume strictly positive correlation, $\Sigma_{12} = \Sigma_{21} > 0$. For a high enough return on one of the assets compared to the other, the optimal weight in this asset is positive, and the other is negative.

2. Assume no correlation between the assets, $\rho = 0$. The optimal weight will not include a shorted position. This is also true in higher dimensions since the variance matrix is a monomial matrix having non-negative entrances in its inverse. Furthermore, since nothing is subtracted, the sum of the weights invested in the risky assets is higher than in the first case.

3. Assume strictly negative correlation, $\Sigma_{12} = \Sigma_{21} < 0$. The minus signs cancel, the weights are positive, and the sum of the weights invested in the risky assets is higher than in the two other cases.

The complication of shorting an asset becomes clear when using Proposition 4.1 on Equation (4.5). Assume $\rho > 0$ and that the constrained optimal real pension product, $\hat{\boldsymbol{w}}$, with no short sales, is the optimal solution. The normal equations are given by

$$\frac{1}{\gamma} = \left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1} \boldsymbol{A}^T\hat{\boldsymbol{w}} \quad \text{for} \quad \boldsymbol{A} = k \begin{pmatrix} \alpha_1\Sigma_{22} - \alpha_2\Sigma_{21} \\ \alpha_2\Sigma_{11} - \alpha_1\Sigma_{12} \end{pmatrix}.$$

The first part is always positive

$$\left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1} = k^{-2}\left[(\alpha_1\Sigma_{22} - \alpha_2\Sigma_{21})^2 + (\alpha_2\Sigma_{11} - \alpha_1\Sigma_{12})^2\right]^{-1} \geq 0.$$

The second part can, for the reasons described in Case 1, have negative values when assuming a real pension product with $\hat{w}_1, \hat{w}_2 \geq 0$ is optimal

$$\boldsymbol{A}^T\hat{\boldsymbol{w}} = k\left[(\alpha_1\Sigma_{22} - \alpha_2\Sigma_{21})\,\hat{w}_1 + (\alpha_2\Sigma_{11} - \alpha_1\Sigma_{12})\,\hat{w}_2\right]. \tag{4.6}$$

The smaller $\boldsymbol{A}^T\hat{\boldsymbol{w}}$ is, the larger γ will be. Two acknowledgeable forces are making $\boldsymbol{A}^T\hat{\boldsymbol{w}}$ smaller than it would be with the theoretically optimal solution $\tilde{\boldsymbol{w}}$. First, an optimal shorted position of asset i in Equation (4.6) creates a negative term since $\hat{w}_i \geq 0$. Second, the optimal geared position in asset $j \neq i$ creates a smaller term when assuming $\hat{w}_j$ to be optimal because $\hat{w}_j \leq \tilde{w}_j$. Furthermore, if the forces mentioned above make $\boldsymbol{A}^T\hat{\boldsymbol{w}} < 0$, then the risk aversion is negative, which is undefined.

△

The example shows that the optimal weight in a risky asset can be negative for $\rho > 0$. On the other hand, if assets are negatively correlated, more is invested in the risky assets, potentially more than 100%, depending on the Sharpe ratio. These two elements make the optimal asset allocation differ from the real pension allocation. Therefore, the risk aversion can be very high or even negative when assuming a constrained-optimal real pension product to be optimal. In Section 5, we find the risk aversion for specific life-cycle products and conclude these issues when comparing them to the literature. The complications lead to ruling out the first approach and instead considering the following approach.

4.1.2 The one-fund approach

Consider a market with life-cycle pension products varying in risk profiles. Let the real pension allocation be the optimal asset allocation for the pension-saver. The one-fund approach is based on the following assumption about the investor's real asset allocation.

Assumption 4.3. *Assume that the real asset allocation is the only risky fund of the market.*

Assumption 4.3 allows us to consider a risky fund making the theoretical framework in Section 3.3 directly applicable. The assumption violates Merton's mutual funds theorem since any investor should be indifferent between investing in the risky fund or a linear combination of the n assets. This is not true for any investor since there are different risk profiles, and only 100% of a savings account can be invested. In other words, the risky fund is **not** independent of the investor's risk aversion. However, an investor **has** chosen to invest 100% in a pension product instead of the n risky assets. Therefore, if we pretend that the pension-saver lives in a world where she can only choose the specific pension product invested in, then the pension product is independent of risk aversion, therefore not violating Merton's mutual funds theorem. Since this is not exactly the case, it is not apparent that different pension products' varying risk profiles make the risk aversions comparable or even correct. The following mathematical introduction provides more intuition about the smoothness of this approach. Further justification about comparability and validity is shown in the numerical study in Section 5.

The process of deriving the risk aversion starts by recalculating the expected return and volatility, $\tilde{\alpha}(t)$ and $\tilde{\sigma}(t)$ today for $t \in (0, N)$. As the risky fund depends on the specific asset allocation, we have to calculate the above parameters for all life-cycle pension products. Since we assume that the pension product is optimal, there is a tilde on the expected return and volatility of the pension asset allocation. We assume 100% is invested in the risky fund, i.e., $\tilde{\pi} = 1$ for all $t \in (0, N)$. Then for a specific life-cycle pension product, the estimated risk aversion today is given by

$$\hat{\gamma} = \frac{\tilde{\alpha}(t) - r}{\tilde{\sigma}(t)^2} \frac{X(t) + X_l^{\mu}(t)}{X(t)}. \tag{4.7}$$

Due to $\tilde{\pi} = 1$, the focus has switched from considering how much to invest in a risky fund to assessing the risk profile of a risky fund such that it is optimal for the investor to have everything invested in it. We are used to the idea that an increased Sharpe ratio and a high amount of human capital compared to current wealth give a higher optimal weight in a risky fund. Now fixating the investment in the risky asset to 100% and assuming it is optimal, an increase in Sharpe ratio or human capital compared to current wealth leads to a higher risk aversion.

Remark 4.4. *When we assume the real asset allocation is the risky fund, we are not modeling the wealth process using the fixed physical market parameters, α, and σ. Instead, we are using the mappings $t \mapsto \tilde{\alpha}(t)$ and $t \mapsto \tilde{\sigma}(t)$, which are time-dependent, as they are the parameters of the risky fund to time $t \in (0, N)$. In Section 5, the time-dependent parameters of the risky fund are implemented in an Euler scheme.*

Different pension products give rise to various risk aversions by determining the relationship between the expected return and the risk of the portfolio, $\frac{\tilde{\alpha}(t)-r}{\tilde{\sigma}(t)^2}$. The fraction $\frac{X(t)+X_l^{\mu}(t)}{X(t)}$ should, for an optimal asset allocation, make the risk aversion constant despite the portfolio changes when approaching retirement. This might not be the case for the sub-optimal asset allocations in Example 3.12. For the sub-optimal asset allocations in Example 3.12, and the constrained optimal asset allocation in Example 3.11, it is especially not the case for a pension-saver in the gearing phase. This is because the pension-saver invests less risky than optimal, and therefore Equation 4.7 overestimates the risk aversion. It is therefore left to consider how to estimate the risk aversion for this group of pension-savers.

Suppose the pension products are, in fact, optimal by Theorem 3.8. In that case, we could calculate the risk aversion with $\frac{\tilde{\alpha}(T)-r}{\tilde{\sigma}(T)^2}$ at retirement, T, as there are no more premiums. From Example 3.12, it is clear that the pension products in the market are sub-optimal since the glide paths are age-dependent and not depending on the development of the savings accounts. The best risk aversion estimate for a pension-saver in the gearing phase is when the de-escalation phase begins. We suggest three age-dependent investment strategies inspired by [Khemka et al., 2021] that project the fraction $\frac{X(t)+X_l^{\mu}(t)}{X(t)}$ in different ways to make Equation (4.7) age-dependent when considering the risk aversion over time.

Consequently, it is possible to calculate an estimate of the risk aversion at all future time points. Therefore, we are interested in estimating a snapshot of the risk aversion today but are forced to be satisfied with a future time point for investors in the gearing phase. As a consequence of the sub-optimality of the real pension allocation and the projection strategy, Equation (4.7) does not provide a fixed estimate for the risk aversion but might vary over time. For the projection strategies, we consider $t \in (0, N)$ where t is not necessarily today.

Recall the constrained optimal investment strategy from Example 3.11 given by

$$\pi^{(0)}(t, X(t)) = \min\left(1, \tilde{\boldsymbol{\pi}}\right),$$

where $\tilde{\boldsymbol{\pi}}$ is the optimal investment strategy from Theorem 3.8. Our approach differs from

[Khemka et al., 2021] as we do not project the wealth process, X, using $\pi^{(0)}$. We use the real asset allocation, which we assume to be optimal. Therefore denote the real asset allocation by $\hat{\pi}$, such that the wealth process in the following is given by $X^{\hat{\pi}}$.

The first investment strategy we use from [Khemka et al., 2021] is given by

$$\pi^{(1)}(t) = \frac{\tilde{\alpha}(t) - r}{\tilde{\sigma}(t)^2} \frac{1}{\gamma} \mathbb{E}\left[\frac{X^{\hat{\pi}}(t) + X_l^{\mu}(t)}{X^{\hat{\pi}}(t)}\right].$$

We isolate the risk aversion for $\pi^{(1)}(t) = 1$ and use Monte Carlo simulation

$$\gamma^{(1)}(t) = \frac{\tilde{\alpha}(t) - r}{\tilde{\sigma}(t)^2} \frac{1}{H} \sum_{i=1}^{H} \frac{X_i^{\hat{\pi}}(t) + X_l^{\mu}(t)}{X_i^{\hat{\pi}}(t)}, \tag{4.8}$$

where H is the number of simulations, and the subscript on $X^{\hat{\pi}}$ denotes the sample paths of the savings account. In Sections 4.2 and 4.4, it is relevant to keep track of the wealth process, $X^{\hat{\pi}}$ in our Euler scheme. Therefore when using $\gamma^{(1)}$ we also use the sample paths of $X^{\hat{\pi}}$ to determine the Monte Carlo simulated wealth process

$$X^{(1)}(t) = \frac{1}{H} \sum_{i=1}^{H} X_i^{\hat{\pi}}(t).$$

The second strategy is given in the following proposition for constant expected return and volatility of the risky fund. By implementing Proposition 4.5 into the same Euler scheme as the first strategy, we can let the market parameters depend on time as described in Remark 4.4.

Proposition 4.5. *Assume $\mu^* \in \mathcal{L}_2$ and $\alpha, \sigma \in \mathbb{R}_+$ to be the expected return and the volatility of the risky fund. Let the wealth process for $t \in (0, N)$ be given by*

$$dX(t) = \pi(\alpha - r)X(t)dt + (X(t)(r + \mu^*(t)) + l)dt + \pi\sigma X(t)dW(t),$$

with $\lim_{t\downarrow 0} X^{\tilde{\pi}}(t) = x_0$. Let the optimal investment strategy be given by

$$\tilde{\pi}(t, X(t)) = \frac{\alpha - r}{\sigma^2} \frac{1}{\gamma} \frac{X(t) + X_l^{\mu}(t)}{X(t)},$$

and the sub-optimal weight in the risky fund be given as

$$\pi^{(2)}(t) = \frac{\alpha - r}{\sigma^2} \frac{1}{\gamma} \frac{\mathbb{E}[X^{\tilde{\pi}}(t) + X_l^{\mu}(t)]}{\mathbb{E}[X^{\tilde{\pi}}(t)]},$$

where X_l^{μ} is the human capital. Using the one-fund approach, then if 100% is held in the risky fund

$$\gamma^{(2)}(t) = \frac{t(\alpha - r)^2}{\left(LW\left(\frac{X_l^{\mu}(t)t(r-\alpha)e^{-t\alpha - \bar{\mu}^*(t)}}{(x_0 + X_l^{\mu}(0))}\right) - rt + \alpha t\right)\sigma^2}, \tag{4.9}$$

for $\bar{\mu}^*(t) = \int_0^t \mu^*(s)ds$*, and where* LW *is the Lambert W function defined for*

$$\frac{X_l^\mu(t)t(r-\alpha)e^{-t\alpha-\bar{\mu}^*(t)}}{(x_0 + X_l^\mu(0))} > -e^{-1}.$$

Proof. By Remark 3.9

$$\begin{aligned}\pi^{(2)}(t) &= \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{\mathbb{E}[X^{\tilde{\pi}}(t)] + X_l^\mu(t)}{\mathbb{E}[X^{\tilde{\pi}}(t)]} \\ &= \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{(x_0 + X_l^\mu(0))\, e^{\left(r+\frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t+\bar{\mu}^*(t)}}{(x_0 + X_l^\mu(0))\, e^{\left(r+\frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t+\bar{\mu}^*(t)} - X_l^\mu(t)},\end{aligned}$$

where $\bar{\mu}^*(t) = \int_0^t \mu^*(s)ds$. Let $\pi^{(2)}(t) = 1$ for all $t \in (0, N)$, take the inverse and reorganize to get

$$\frac{X_l^\mu(t)}{(x_0 + X_l^\mu(0))} = \left(1 + \frac{r-\alpha}{\sigma^2}\frac{1}{\gamma}\right) e^{\left(r+\frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t+\bar{\mu}^*(t)}.$$

Multiply with $t\,(r-\alpha)\,e^{-\alpha t-\bar{\mu}^*(t)}$ on both sides in order to recognize the function $f(x) = xe^x$. Then use the Lambert W function, [Wikipedia, 2022], given by $LW(xe^x) = x$.

$$\begin{aligned}\frac{X_l^\mu(t)t\,(r-\alpha)\,e^{-\alpha t-\bar{\mu}^*(t)}}{(x_0 + X_l^\mu(0))} &= \left(r - \alpha + \frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right) te^{\left(r-\alpha+\frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t} \\ &\Longleftrightarrow \\ LW\left(\frac{X_l^\mu(t)t(r-\alpha)e^{-t\alpha-\bar{\mu}^*(t)}}{(x_0 + X_l^\mu(0))}\right) &= \left(r - \alpha + \frac{1}{\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right) t \\ &\Longleftrightarrow \\ \gamma^{(2)}(t) &= \frac{t(\alpha-r)^2}{\left(LW\left(\frac{X_l^\mu(t)t(r-\alpha)e^{-t\alpha-\bar{\mu}^*(t)}}{(x_0+X_l^\mu(0))}\right) - rt + \alpha t\right)\sigma^2}.\end{aligned}$$

The Lambert W function is defined for

$$\frac{X_l^\mu(t)t(r-\alpha)e^{-t\alpha-\bar{\mu}^*(t)}}{(x_0 + X_l^\mu(0))} > -e^{-1}.$$

□

The second strategy is based on the savings account with an optimal investment strategy, which we know the real pension products do not follow. We evaluate this observation in the numerical study. The projection of the wealth process is according to Remark 3.9, given by

$$X^{(2)}(t) = (x_0 + X_l^\mu(0))\, e^{\left(r+\frac{1}{\gamma}\left(\frac{\tilde{\alpha}(t)-r}{\tilde{\sigma}(t)}\right)^2\right)t+\bar{\mu}^*(t)} - X_l^\mu(t),$$

which we use in Sections 4.2 and 4.4.

Lastly, we will use the non-stochastic strategy of letting the savings account accumulate with the risk-free expected return of the risky fund and the mortality rate, i.e., with a yearly return of $e^{\tilde{\alpha}(t)+\mu^*(t)}$. We will denote this as the third projection strategy. As we disregard the volatility of the assets, the third strategy has a discrepancy error compared to the first strategy. However, the second and third strategies can be executed without simulation, granting a significantly lower computing time.

Figure 4.1 shows the difference in the glide path of the strategies from [Khemka et al., 2021] based on the parameters of Example 3.11. The strategies have therefore fixed market parameters, and we have applied the mapping $x \mapsto \min(1, x)$. As introduced in this thesis, we have added mortality risk in all three strategies.

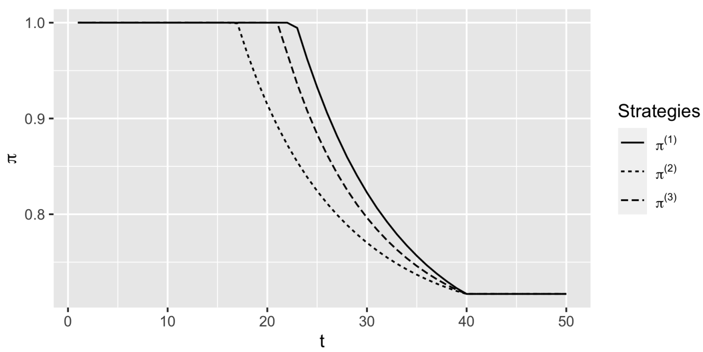

Figure 4.1: *Weight in the risky asset, y-axis, over time, x-axis, for 10,000 simulations of the wealth processes based on strategy* $\pi^{(1)}$, $\pi^{(2)}$ *and* $\pi^{(3)}$.

4.2 Real estate

A real estate investment can consist of a mortgage, a bank loan, and real estate. In this section, we do not consider the bank loan but assume a mortgage finances the entire investment. Real estate investments are riskier than a diversified portfolio, as all the eggs are in one basket. The additional risk is an excellent opportunity to align the investment strategy with the optimal investment strategy, as there is an opportunity for high risk and gearing for the young investor.

In this subsection, we translate the mortgage and real estate investment into the asset classes provided by *Rådet for Afkastforventninger* and embed the problem in the stochastic control theoretic framework.

We simplify the loan to be a mortgage, which can be replicated in the first asset class *government and mortgage bonds*. It is possible to replicate a bank loan in a risk-free asset. However, we want to keep the risk-free return rate fixed for comparability between individuals and categories, excluding different coupon rates. The real estate investment is to be found in the ninth asset class *real estate*. Notice that the market parameters provided by *Rådet for Afkastforventninger* are averages for the general market. For private real estate investments, there is information about the expected return and volatility depending on the size, condition, and location of the real estate. There are as well different time horizons and mortgage interest rates that could be considered.

We determine the weights in *government and mortgage bonds* and *real estate* to calculate the parameters of the real estate portfolio, constituting a risky fund. To this end, let the wealth process X denote the free capital in the real estate, i.e., the difference between the market value of the real estate and the mortgage. Let the remaining debt to time t be defined as the adapted stochastic variable $\psi(t)$, and the market price of the real estate to time t to be the adapted stochastic variable $\phi(t)$. A restriction to the stochastic control theoretic setup defined in Section 2 is that the wealth process may not be negative. Therefore we cannot estimate the risk aversion today in the case of a decline in the real estate market that makes the free capital negative. In Section 4.4, we discuss how to handle a negative free capital to find the certainty equivalent balance upon retirement.

The wealth process does not naturally conform with the setup, including mortality risk from Section 3.3.4. The mortgage bank's only objective is that the obligations are met; hence, it does not benefit from the borrower's death. There are, however, equity release schemes that fit the theory under certain circumstances. Assume that the investor buys a home reversion plan, [The Guardian, 2013], starting at retirement. The investor sells 100% of her home but can continue living there and is guaranteed some percentage of the free capital as benefits until she dies. Upon death, the home reversion company inherits the free capital. We go through the trouble of defining this uncommon product because we want to use the same wealth process with mortality in all categories. This helps us when we define a total wealth process of all categories, which we study in Section 4.4.

We continue to model the free capital with mortality risk given by

$$dX(t) = (r + \boldsymbol{\pi}(\alpha - r) + \mu^*(t))X(t)dt + \boldsymbol{\pi}\sigma X(t)dW(t) + l(t)dt, \tag{4.10}$$

for $t \in (0, N)$ and where the premium, l, denote the payment rate of the mortgage. Assume the real estate is purchased and the debt is established to time $t + dt \in (0, N)$. Then we have $X(t) = 0$, and $X(t+dt) > 0$. By the one-fund approach, we assume the real estate investment to be the risky fund, and we now proceed to calculate the risky fund's weights in the asset classes. The weight placed in *government and mortgage bonds* is given by

$$\tilde{w}_1(t, \psi(t), X(t)) = -\frac{\psi(t)}{X(t)},$$

and the weight in *real estate* is given by

$$\tilde{w}_9(t, \phi(t), X(t)) = \frac{\phi(t)}{X(t)}.$$

The free capital is therefore invested in the market by shorting *government and mortgage bonds* and gearing *real estate*. The rest of the asset classes have weight zero, such that the sum of the weights is one. The risky fund is therefore given by

$$\tilde{\boldsymbol{w}}_{\text{Real estate}} = \begin{pmatrix} \tilde{w}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{w}_9 & 0 \end{pmatrix}^T. \tag{4.11}$$

Unlike pension products, there is no short-sale constraint in the real estate category. Hence we do not need to project the wealth process to estimate the risk aversion. As in pension, we are interested in a snapshot of the risk aversion today. However, we use the projection setup from the pension category to show the glide paths of different real estate investments and to determine a good projection strategy for the total wealth in Section 4.4.

If we assume that the present value of all future installments approximately gives the present value of the size of the loan, then we can approximate

$$\tilde{w}_1 \approx -\frac{X_l^\mu(t)}{X(t)}, \tag{4.12}$$

and

$$\tilde{w}_9 \approx \frac{X(t) + X_l^\mu(t)}{X(t)}, \tag{4.13}$$

where we assume that the mortgage is a life annuity with an instantaneous interest rate equal to the coupon rate of the mortgage bond, i.e., the first asset class. For simplicity, we assumed the mortgage is a life annuity, but one might as well define an annuity without mortality. The real estate portfolio defined by Equation (4.12) and (4.13) is still stochastic but recognizable from the strategies presented in Section 4.1. As in Section 4.1, we wish to investigate which sub-optimal strategy gives the best estimate of the wealth process where computing time is the main trade-off.

4.3 Free funds

The pension and the real estate asset allocations are, to a great extent, inflexible, whereas free funds can be invested more freely. Therefore, the free funds category is considered a buffer, where the investor can take more or less risk to adjust her financial risk across all three categories. The free funds category does not necessarily follow a life-cycle investment strategy but should do in combination with the other categories. The risk aversion in this category is not considered in the numerical study in Section 5.

The stochastic control theoretical framework is the same as in Section 4.1. For the setup with mortality risk, we again have to consider a pension-like product where a deferred life annuity is bought, and the insurance company is the sole inheritor of the wealth process upon death.

4.4 Mutual category

The investor's general wealth can be put in terms of a pension portfolio that can vary in risk profile, a non-diversified real estate investment, and the investments of free funds as a risk buffer. First, we can compare an investor's estimated mutual risk aversion today with others. Secondly, we can use it to calculate certainty equivalents as explained in Section 4.5.

In this subsection, we consider the union of the separate financial categories. We provide the setup to estimate the risk aversion today and to project the total wealth and mutual risk aversion at future time points.

For a young individual, it seems more optimal with the geared real estate position accompanied by a more diversified position in the pension allocation than considering each category separately. Therefore we do not project the risk aversion of an investor with a pension asset allocation in the gearing phase. However, the methods in the separate categories are relevant when joining them. As discussed in the previous sections, we continue under the setup with mortality risk with the necessary assumptions presented in each category.

The challenge is that we know from the real estate category that the free capital process can become negative in reality but not in our setup. We, therefore, use the savings account and the free funds as buffers such that the total wealth process is less likely to be zero. Let $t \in (0, N)$ be today, and the total wealth process be the sum of each category, i.e.,

$$X(t) = X_{\text{Pension}}(t) + X_{\text{Real estate}}(t) + X_{\text{Free funds}}(t). \tag{4.14}$$

The weights in the ten asset classes are given as the sum of the rescaled 10×1 weight vectors from each category

$$\tilde{\boldsymbol{w}} = \frac{\tilde{\boldsymbol{w}}_{\text{Pension}} X_{\text{Pension}}(t)}{X(t)} + \frac{\tilde{\boldsymbol{w}}_{\text{Real estate}} X_{\text{Real estate}}(t)}{X(t)} + \frac{\tilde{\boldsymbol{w}}_{\text{Free funds}} X_{\text{Free funds}}(t)}{X(t)}, \tag{4.15}$$

where the $\tilde{\boldsymbol{w}}_{\text{Pension}}$ is the pension allocation, $\tilde{\boldsymbol{w}}_{\text{Real estate}}$ is given by Equation (4.11) and $\tilde{\boldsymbol{w}}_{\text{Free funds}}$ is the distribution of free funds in the asset classes. Therefore, the allocation in the risky fund is given by $\tilde{\boldsymbol{w}}$, and the risk aversion can be estimated with the one-fund approach.

In Section 4.5, we need the projection of the total wealth process. Initially, we consider projecting each wealth process separately, and then for all $t \in (0, N)$ use Equations (4.14) and (4.15) to estimate the risk aversion. This procedure inherits the issue from the real estate category, where the probability of the free capital hitting zero for a small wealth process is high. We can not model a negative free capital process. Therefore suppose we let the free capital be equal to zero and wait for the next mortgage installment. In that case, our setup does not reveal a low enough risk aversion, but prohibiting the free capital from recovering from zero with the next mortgage installment does the opposite. The latter is what we know as bankruptcy, where the real estate would be sold to cover the obligations to the mortgage bank.

Instead, we find the weight in real estate residually, so we are less likely to model a negative free capital process. Let $t \in (0, N)$ be today, $\phi(t)$ be the market value of the real estate, and $X_l^{\mu}(t)$ be a life annuity of the discounted value of all future payments on the mortgage. Then today, the total wealth process is given by

$$X(t) = X_{\text{Pension}}(t) + X_{\text{Free funds}}(t) + \phi(t) - X_l^{\mu}(t).$$

The risky fund is therefore given by Equation (4.15) to time t. The parameters of the risky fund α_{mutual} and σ_{mutual} can be calculated and the total wealth process is projected in the interval $[t, t + dt]$ with the dynamics

$$dX(t) = (\alpha_{\text{mutual}} + \mu^*(t))X(t)dt + \sigma_{\text{mutual}} X(t)dW(t) + l_{\text{mutual}}dt, \tag{4.16}$$

where l_{mutual} is a fixed payment rate of premium to the savings account, money to invest in free funds, and the mortgage installment. Notice that Equation (4.16) is the usual wealth process dynamics as in Equation (4.10) but with the assumption that $\tilde{\boldsymbol{\pi}} = 1$.

We now have the savings account, the free funds account, $X(t + dt)$, and the respective weights in the market of each category except the weight in the ninth asset class (real estate) in the real estate category. We can residually calculate this weight to time $t + dt$ to be

$$\tilde{\boldsymbol{w}}_{\text{Real estate}}(t+dt)^T \boldsymbol{e}_9 = 1 - \Bigg(\frac{\tilde{\boldsymbol{w}}_{\text{Pension}}(t+dt) X_{\text{Real estate}}(t+dt)}{X(t+dt)} + \frac{\tilde{\boldsymbol{w}}_{\text{Free funds}}(t+dt) X_{\text{Free funds}}(t+dt)}{X(t+dt)} + \frac{\tilde{\boldsymbol{w}}_{\psi} X_l^{\mu}(t+dt)}{X(t+dt)} \Bigg)^T \boldsymbol{e},$$

where $\boldsymbol{e}_9$ is the zero vector with one in the ninth entrance, and

$$\tilde{\boldsymbol{w}}_{\psi} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

This finalizes the projection procedure as the next iteration begins from this point. We only consider the pension and real estate categories in the numerical study. Then the Euler scheme from the other categories can be used to update the parameters of the savings account and the total wealth.

4.5 Certainty equivalents

So far, we can estimate three risk aversions and a fourth mutual risk aversion. However, the risk aversion quantity does not make sense beyond comparing categories. Still, we cannot explain the effect of having one unit larger or smaller risk aversion. Instead, we can take the optimal value function for the accumulation phase and apply the inverse utility function to get the expected certainty equivalent of terminal wealth. The quantity is now on a monetary scale instead of a utility-scale based on the investor's risk aversion. Recall from Example 3.13 that the certainty equivalent is the amount that leaves the investor indifferent between adopting the risky strategy or receiving a guaranteed amount. Therefore, the investor aims to maximize the expected certainty equivalent of terminal wealth. How much the estimated risk aversions in the three financial categories differ might be more comprehensible for an investor. Therefore a calibration exercise of the risk aversion is more apparent as the investor can understand the monetary change by taking other financial choices that lead to higher or lower certainty equivalents.

In [Khemka et al., 2021], it is suggested to calculate the Certainty Equivalent Balance (CEB) at retirement given by

$$CEB = \left((1-\gamma) \left[\frac{\sum_{i=1}^{H} \frac{1}{1-\gamma} X_i(T)^{1-\gamma}}{H} \right] \right)^{\frac{1}{1-\gamma}}, \tag{4.17}$$

where H is the number of simulations and i denotes the sample paths of the stochastic account X. It is based on a Monte Carlo simulation rather than the analytical approach used to find the expected certainty equivalent of terminal wealth. The CEB is the amount that leaves the investor indifferent between adopting the risky strategy or receiving the guaranteed amount when retiring. Equation 4.17 is, therefore, the average utility of the account at retirement in each sample path transformed into the monetary scale of certainty equivalents. We can use the estimated mutual risk aversion to investigate changes in CEB over different financial actions. It is left for the numerical study to clarify which financial actions maximize CEB. Since the estimated risk aversion is not constant as we assume the real risk aversion is, we consider the mean risk aversion over time when calculating CEB in the numerical study.

5 Numerical study

The numerical study aims to empirically investigate the market assumptions, the approach proposed in Section 4.1, and the extension to real estate and a mutual financial category. We distinguish between estimating an investor's risk aversion today and estimating future risk aversions with the projection strategies. The latter is used for pension-savers in the gearing phase, to check the assumption of constant risk aversion, and to determine the certainty equivalent balance at retirement.

The first three subsections describe the setup of the numerical study. We introduce and discuss the market parameters introduced by *Rådet for Afkastforventninger*, the model estimating risk aversion, and the pension data we study.

In the last three subsections, we present estimates of risk aversions in the pension category for different market parameters published by *Rådet for Afkastforventninger* and argue which projection strategy to use for young pension-savers in the gearing phase. The projection strategy is also used to check the assumption of an investor having the same risk aversion over time. Monotonic variations in risk aversion could be caused by the projection strategy or be a measure of the sub-optimality of the investment strategy. Secondly, we study how well the real estate category conforms with the principles of life-cycle investments when applying the setup from Section 4.2. Lastly, we examine the total wealth of the pension and real estate categories and introduce financial advice based on maximizing the certainty equivalent balance at retirement.

5.1 Rådet for Afkastforventninger

In response to the emergence of unit link pension products, there were, amongst other things, suggested a standard risk scale for Danish pension companies. The standard risk scale is based on confidence intervals for savings account projections one year forward leading to a *risk number*. This thesis offers another way of describing risk to the pension-saver. The savings account projections are based on market parameters supplied by *Rådet for Afkastforventninger*, [Rådet for Afkastforventninger, 2022], for the market divided into ten asset classes. The asset classes, expected returns, volatility, and correlations are given in Appendix A.4 and constitute the input to the market from Section 3.1.

Rådet for Afkastforventninger revises the market parameters every half year and divides the parameters into three intervals: The next $[1, 6)$, $[6, 11)$ and $[11, \infty)$ years ahead. In the last interval, the ten classes are merged into three classes. Before 2023 there were only two merged asset classes, bonds, and stocks, for the long-term projections. From 2023, the three merged asset classes are given by: Bonds (asset class 1-4), stocks (asset class 5-6), and illiquid assets (asset class 7-10). Illiquid assets and stocks have the same expected return, but illiquid assets have lower volatility. The explanation is that the investor gains an illiquidity premium. The market assumptions in this thesis only comply with the ratio between risk and return and not conventions like illiquidity. The weight of illiquid assets in a life-cycle pension product is typically not dominating. Given that companies make the same use of illiquid assets, the volatility advantage will be the same over different products. However, it will have a direct effect on the real estate category.

We know from Section 4.1 that correlations in the market can provoke short-selling in the optimal asset allocation, a feature the pension products do not allow. The numerical effect of the first approach in Section 4.1 is presented in Section 5.4. We continue with the market suggested by *Rådet for Afkastforventninger* since we believe that the one-fund approach provides good results and the asset classes fit the needs in the financial categories; pension, real estate, and free funds.

5.2 The pension data

We consider 5236 policies with a deferred life annuity distributed over the low, moderate, and high-risk profiles illustrated in Figure 5.1. The life-cycle products follow the characteristic glide paths from Example 3.12, where the weight in the risky asset is the sum of the stocks and illiquid asset classes, i.e., asset class 5-10.

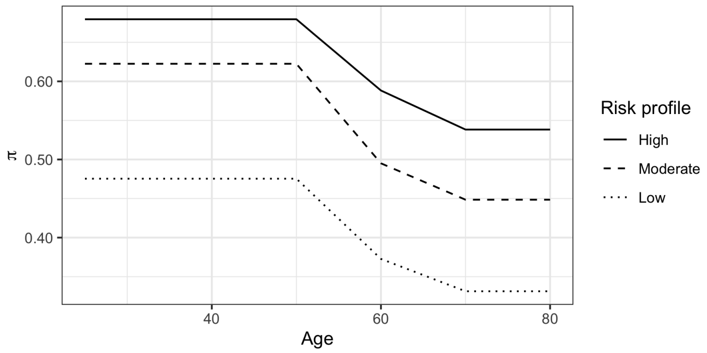

Figure 5.1: *Low, moderate, and high-risk life-cycle products for an investor with retirement at age 65. The y-axis is the weight in stocks and illiquid asset classes (asset class 5-10).*

The asset allocation is given 30, 15, and 5 years before and after retirement. The de-escalation of risk begins 15 years before retirement for all risk profiles.

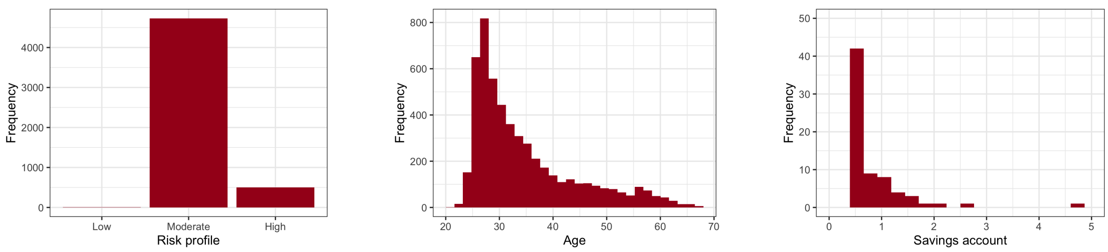

Figure 5.2: *Empirical distributions of risk profiles, ages, and savings accounts in millions DKK for the 5236 policies with life-cycle pension products.*

In Figure 5.2, there is a great overweight of moderate risk profiles, whereas there are only six with low and 500 with high-risk profiles. The life-cycle pension products are new in general, which is clear in Figure 5.2, as the savings accounts are small, and the age distribution favors the young. Therefore, the data lack individuals with low and high-risk profiles, and individuals who have been saving up with a life-cycle product for a longer time. The pension-savers in the de-escalation phase might have transferred their savings account from another product, creating a small group where we can estimate risk aversions. Many pension-savers in the age 30–50 create noise because they start late in life in a life-cycle product. Therefore we reduce the policies we study to 2321 policies where the pension-savers in the gearing phase all are under the age of 30.

The human capital term is affected by the time until retirement. According to [Styrelsen for Arbejdsmarked og Rekruttering, 2021], it is assumed that the retirement age is increasing due to lifetime improvements. Life-cycle products, as the ones considered, that begin to de-escalate at a specific time before retirement makes the gearing phase longer if the starting age is assumed fixed. Whether it is optimal for a young individual to invest in a pension is discussed in [Finans, 2022]. From a utility point of view, the pension payments early in life might give more utility in a geared real estate investment, which is considered later.

5.3 The model

The model is based on an Euler scheme that updates the expected return and volatility of the risky fund every year. We use the real pension allocations from Figure 5.1, where we use linear interpolation between the time points to get the yearly allocation. The interest rate is assumed to be $r = 0.02$, the average of the first asset class, *government and mortgage bonds* for the first ten years. The age, premium, savings account, and risk profile are given in the pension data, where the units are year and DKK. Finally, we assume the market parameters provided by *Rådet for Afkastforventninger* and the mortality risk for a woman of age y from the G82 basis, [P+, 2022],

$$\mu_y^*(t) = 0.0005 + 10^{5.728+0.038(y+t)-10}.$$

The additional input in the real estate category is two of the three quantities measured today: The real estate market value, the mortgage, and the free capital.

For the projection strategy $\gamma^{(1)}$, there are implemented $H = 10,000$ simulated sample paths of the wealth process. Without the premium and consumption processes, the wealth process is a GBM. In each step of the Euler scheme, the wealth process is calculated as a GBM without a premium. The premium is then added to the starting value each year. The same Euler scheme is used to update the parameters of the risky fund for $\gamma^{(2)}$ and $\gamma^{(3)}$.

5.4 Pension

We show that the first approach from Section 4.1 with the market given by *Rådet for Afkastforventninger* leads to unsatisfactory results. Then we move on to the one-fund method for the retirement, de-escalation, and gearing phase.

5.4.1 The first approach

Using the normal equations, we calculate risk aversions using the short-term market parameters published for the last half year of 2022, 2022Q3-4, and the first half year of 2023, 2023Q1-2. We only show the risk aversion for retired pension-savers with low, moderate, and high-risk profiles. Due to the expected return of zero on the first asset class in 2022Q3-4, we present in Table 5.1 the results for an interest rate of 0% and 0.02%.

$r = 0$

Risk profile	Risk aversion 2022Q3-4	Risk aversion 2023Q1-2
High	157.40	68.88
Moderate	105.95	60.63
Low	74.20	52.41

$r = 0.02$

Risk profile	Risk aversion 2022Q3-4	Risk aversion 2023Q1-2
High	-325.59	79.97
Moderate	-199.65	192.11
Low	-132.46	-229.63

Table 5.1: *The risk aversion for three different risk profiles based on the short-term parameters from the last half of 2022 and the first half of 2023 with different interest rates.*

Table 5.1 shows that without the assumption that all entrances in the physical market parameters should be greater than or equal to the interest rate, as assumed in Section 3.1, the risk aversion can be negative hence undefined. The low-risk profile for $r = 0.02$ and 2023Q1-2 justifies the issues discussed in Example 4.2 where the risk aversion could be negative due to the correlations in the market and the constrained optimal pension allocation. The variation between the risk aversion estimates in 2022Q3-4 and 2023Q1-2 tells us that the first approach is very sensitive to the interest rate. Furthermore, the market parameters of 2022Q3-4 cause the risk aversion to be highest for the riskiest pension profile, which has to do with low expectations of the bond market and is discussed later. Lastly, we have risk aversion quantities larger than in previous studies, usually, in the range of $[0.35, 9]$, see Table 1 in [Conine et al., 2016]. This range can be accomplished with the one-fund approach.

5.4.2 The one-fund approach

Consider the pension-savers who have been retired for at least five years and therefore have a constant asset allocation. Their risk aversions can be calculated by isolating the risk aversion in the Merton constant with the parameters of the risky fund. For the products, in Figure 5.1, we show the risk aversion for the market parameters from 2022Q3-4 and 2023Q1-2 in Table 5.2. In the parameters from 2023Q1-2, there is a significant increase in bond price expectations for the next five years. This is due to the increased interest rates to fight inflation during the second half of 2022. The market, in general, is also expected to give higher returns in the newest market parameters.

Risk profile	Risk aversion 2022Q3-4	Risk aversion 2023Q1-2
High	1.61	2.55
Moderate	1.59	2.75
Low	1.25	3.38

Table 5.2: *The risk aversion for three different risk profiles based on the short terms parameters from the last half of 2022 and the first half of 2023.*

Table 5.2 shows that the estimated risk aversion depends heavily on the market parameters. The low-risk profile calculated from the 2022 parameters leads to the lowest risk aversion, which is counter-intuitive but a sign of the suffering bond market. Therefore, the simple investment strategy of minimizing portfolio risk with bonds has given false security. The risk aversion is generally higher based on the 2023Q1-2 parameters. The interpretation is that Merton's constant implies that the optimal weight increases in a market with a higher Sharpe ratio. However, as the pension allocation is fixed, the investor becomes more risk-averse.

Next, consider pension-savers in the de-escalation phase, which for the products in this dataset range from age 50 until five years after retirement, where retirement is assumed to be at age 65. Since the life-cycle products are new, most pension-savers in this age group did not buy this product at the beginning of their life-cycle. The only pension-savers we can study have their savings account from another source. The risk aversions are given in Figure 5.3, where we have chosen a subset of savings accounts of size as if they have paid premium since they were 25 and had no return on investments.

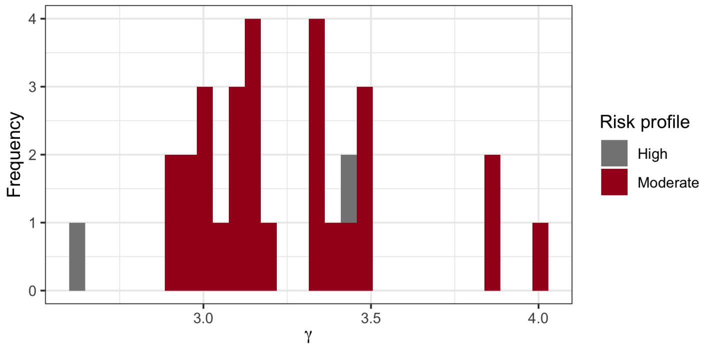

Figure 5.3: *Subset of pension-savers at age 50–70, where the x-axis is risk aversion.*

From Figure 5.3, there are too few with a high-risk profile to conclude a tendency. Still, the moderate risk profiles in this age group tend to have a higher risk aversion than the retired group in Table 5.2. The variability in risk aversion within the same risk profile is a sign of the sub-optimality of the pension product. Assume that the pension-saver's true risk aversion today is lower than the one we estimate when she retires. We could calculate the percentage loss in expected certainty equivalent of terminal wealth for the two risk aversions. By asking about certainty equivalents, we can calibrate the risk aversion and advice on more suitable risk profiles. We do not pursue these calculations any further in this thesis.

The last group to examine is the pension-savers in the gearing phase, where we study the three strategies proposed in Section 4.1, $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$. To this end, we choose a pension-saver from each risk profile approximately the same age and all in the gearing phase; see Table 5.3.

Risk	y	l_{Pension}	X_{Pension}
High	26.92	61,122.36	47,158.47
Moderate	26.92	45,676.92	35,839.46
Low	26.83	60,332.16	29,645.33

Table 5.3: *Three different risk profiles in the gearing phase, where y is the age of the pension-saver in years, $l_{Pension}$ is the yearly premium and $X_{Pension}$ is the savings account today.*

The risk aversions at age 50, where the gearing phase is over, are in Figure 5.4 approximately given by 7.8, 5.4, and 4.6 for the low, moderate, and high-risk profiles. The rest of the lifetime fluctuations are interpreted as sub-optimality of the pension allocation. Especially the low-risk profile has allocations that lead to an unstable risk aversion. In principle, the pension-saver could change risk profile or even pension company over time – or be guided to do so. Risk aversion over time might be a tool to prepare for market changes by changing an investor's wealth allocation or risk profile. For instance, Table 5.2 concluded that the bond market had offered false security with the previous market parameters. Any variation in the strategies late in life does not significantly impact the risk aversion estimates as the account's influence decreases over time as the human capital reaches zero. All projection strategies perform almost equally well. There is however a great computing time advantage in using $\gamma^{(2)}$ or $\gamma^{(3)}$, since there is no need for simulation. This is advantageous when we estimate the risk aversion for the rest of the pension-savers in the gearing phase.

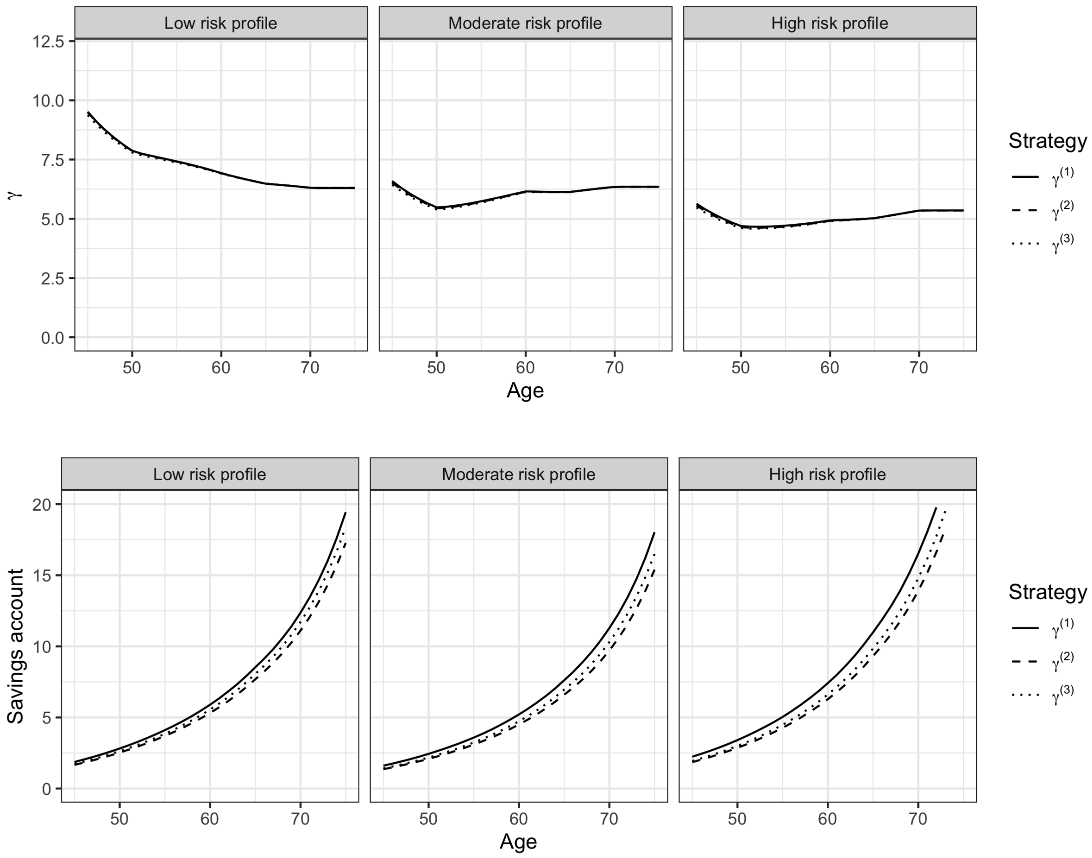

Figure 5.4: *Projections for the three risk profiles where the top row is risk aversion over time, and the bottom row is the savings account in millions over time.*

Regarding the saving accounts in Figure 5.4, we know that there is a discrepancy error on the third strategy making it perform worse than the first. The second strategy has even slower development than the third strategy. Recall we subtract the human capital from the GBM $\{X^{\tilde{\pi}}(t) + X_l^{\mu}(t)\}_{t\geq 0}$ in order to get the savings account only. The savings account performs worse, when letting the GBM develop with the constrained and sub-optimal real asset allocation instead of with the optimal investment strategy. We notice the low-risk profile has a higher premium than the moderate, which is why the account of the low-risk profile outperforms over time. We believe that the first projection strategy is the most precise, but we acknowledge that the third strategy comes close.

Figure 5.5 shows the risk aversions for the three phases, where the gearing phase is estimated with $\gamma^{(3)}$. We see that some of the moderate risk profiles in the right graph in Figure 5.5 are spread around a risk aversion of 3.2, which comes from Figure 5.3, and a risk aversion of 5.4 which are policies from the gearing phase. The variation between the gearing and de-escalation phases leads to considering only the short-term market parameters for the gearing phase. We argue that today's market parameters reflect the circumstances under which the pension-saver has chosen the risk profile. If the market was to change, the pension-saver might also change the risk profile. Choosing to only project with the short-term market parameters grants a more consistent risk aversion, as shown in Figure 5.5 to the left.

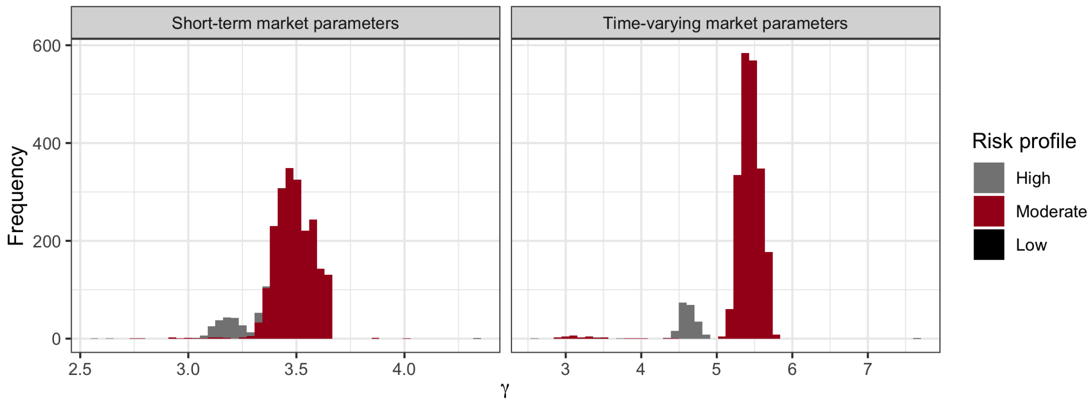

Figure 5.5: *Pension-savers in all three phases, where the risk aversion is estimated with* $\gamma^{(3)}$ *and indicated on the x-axis.*

From the pension category and the pension dataset, we conclude that $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$ are almost equally preferable, but $\gamma^{(1)}$ is most precise and $\gamma^{(3)}$ comes closest but gives a lower computational time. However, we need the time-consuming simulation strategy to calculate certainty equivalent balance (CEB) upon retirement. The most consistent risk aversion over the gearing, de-escalation, and retirement phase is given by using fixed market parameters and considering policies that, within reason, have a savings account of size as if they have had the pension product since they were at most 30 years old. Regarding the best projection of the savings account, we favor the Monte Carlo simulation, which we will keep in mind for the real estate category.

5.5 Real estate

This section studies the nature of different real estate investments in the life-cycle setup. We consider the severity of the free capital going to zero for low starting values, as we can not estimate the risk aversion in this case.

The approximations in Equation 4.12 and 4.13 provide mortgage bonds and real estate weights. We examine the risk aversion of three investments for a pension-saver at age 28 as given in Table 5.4. The table contains an investment with the legally smallest free capital of 5% of the real estate value upon purchase and two investments varying in length of the mortgage. The mortgage is assumed to be a life annuity with a coupon rate of $r = 0.02$.

Investment	t_ψ	l_ψ	ψ_0	$X_{\text{Real estate}}$
1	30	0.071	1.5	1.0
2	30	0.111	2.375	0.125
3	10	0.185	1.5	1.0

Table 5.4: *Three real estate investments for a pension-saver at age 28, where t_ψ is mortgage length in years, l_ψ is yearly mortgage installments, ψ_0 is the mortgage today and $X_{Real\ estate}$ is the free capital today all stated in millions.*

The glide path of the real estate investment is given in Figure 5.6. When the free capital hits zero, we let the weights and the risk aversion be zero and recalculate the weights next year with the next installment. This keeps us from modeling the case of negative free capital; therefore, we also have that the real estate investment is less risky than in reality. The glide paths exhibit steepness when the mortgage is short, or the free capital starts low. The latter situation is evident in the second investment in Figure 5.6 where the steep glide path comes from the high gearing that makes the free capital increase significantly *in expectation*.

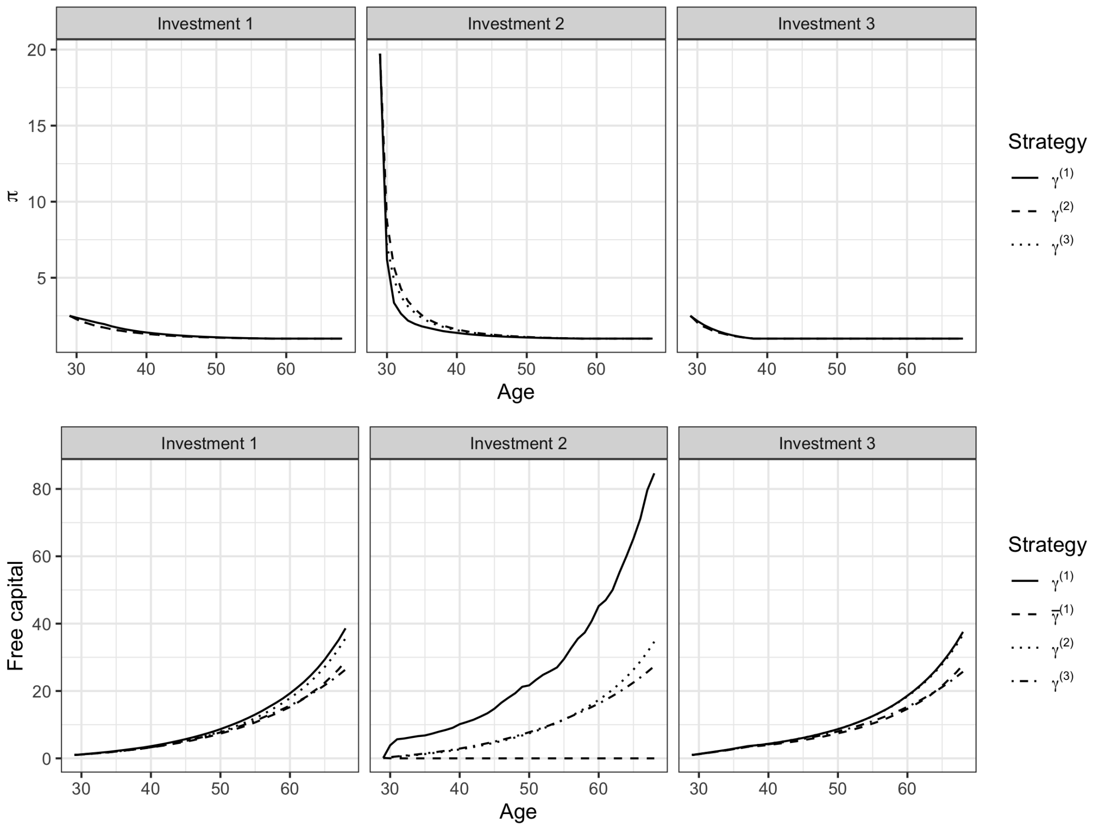

Figure 5.6: *The top row is glide paths for the three investments for a 28-year-old, where the y-axis is the weight in real estate. The second row is the free capital belonging to the investments in the top row, where the y-axis is the free capital in millions. We denote using the median in the Monte Carlo simulation by $\bar{\gamma}^{(1)}$.*

The second row in Figure 5.6 shows the development in the free capital, where we have defined $\bar{\gamma}^{(1)}$ to be the Monte Carlo simulation but where we use the median instead of the mean. In the second investment, the free capital hits zero at age 65 in 71.5% of the 10,000 simulations, which is illustrated as the median is zero for almost all ages. It turns out that either we include extreme values with the mean or end up with what seems to be a bankruptcy situation with the median. The second and third strategies have the advantage that the free capital does not hit zero, and they are computationally faster. The second strategy is based on the optimal investment strategy, and it, therefore, fits the real estate category better than the pension category.

In Figure 5.7 $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$ all converge to the same risk aversion as all investments end with having 100% in real estate. This is the reason we suggested that different locations, sizes, and conditions might affect the expected return and volatility of the individual real estate. Until the risk aversion flattens, it is affected by the time-varying physical market parameters. However, even if these were fixed, there would still be an increasing tendency in risk aversion. We conclude that the real estate investment can act as a life-cycle investment as the variability in risk aversion is not much more significant than in the pension category. Lastly, it is confirmed that at initiation, all real estate investments from Table 5.4 are riskier than a pension product because they lead to a lower risk aversion. However, the greater volatility of the real estate investment early in life might make up for the shorting-constraint in pension, making both investments more optimal if combined.

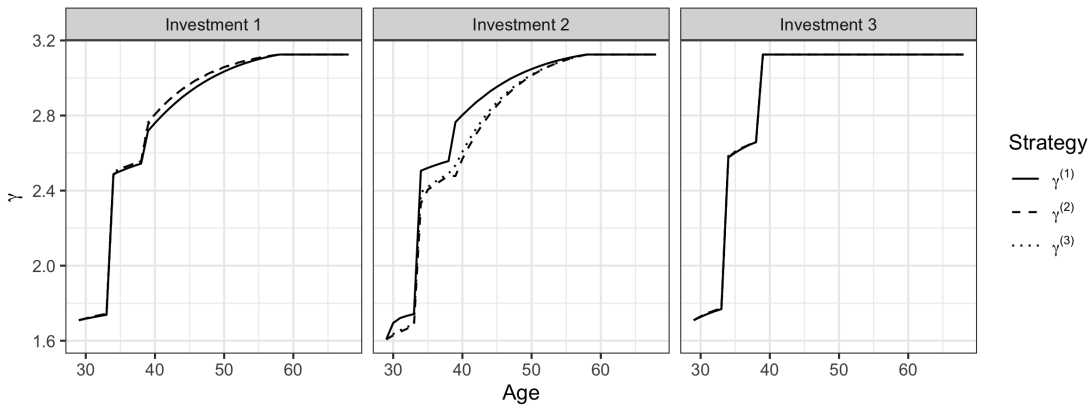

Figure 5.7: *Projected risk aversion estimates with $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$ for three investments for a 28 years old. The y-axis is the estimated risk aversion.*

5.6 The mutual category

With the setup from Section 4.4, we want to calculate the glide paths and risk aversions of a 28-year-old investor with a moderate pension risk profile from Table 5.3 accompanied by the real estate investments from Table 5.4. Afterward, we consider the moderate risk profile with the first real estate investment and use Monte Carlo simulation to calculate the CEB as introduced in Section 4.2.

Due to the free capital tending to hit zero, we consider the setup where the savings account acts as a buffer. Furthermore, as the total wealth process sometimes becomes zero or very close to zero, we use the median in the Monte Carlo simulation instead of the mean when calculating the glide paths and risk aversions. Figure 5.8 shows the simulated glide paths of the weight in asset class 5-10. As in Section 5.5, we have set the weights to zero if the total wealth process hits zero. Despite having the savings account as a buffer, the second investment strategy makes the total wealth process hit zero at age 65 in 50% of the 10,000 simulations. We again see the median in the second investment in Figure 5.8 tends to zero very fast. We conclude that the wealth process still has trouble surviving with a low starting value for the free capital and a small savings account. When using $\gamma^{(2)}$, the savings account develops faster than the free capital giving the first and the third investment a more decreasing tendency.

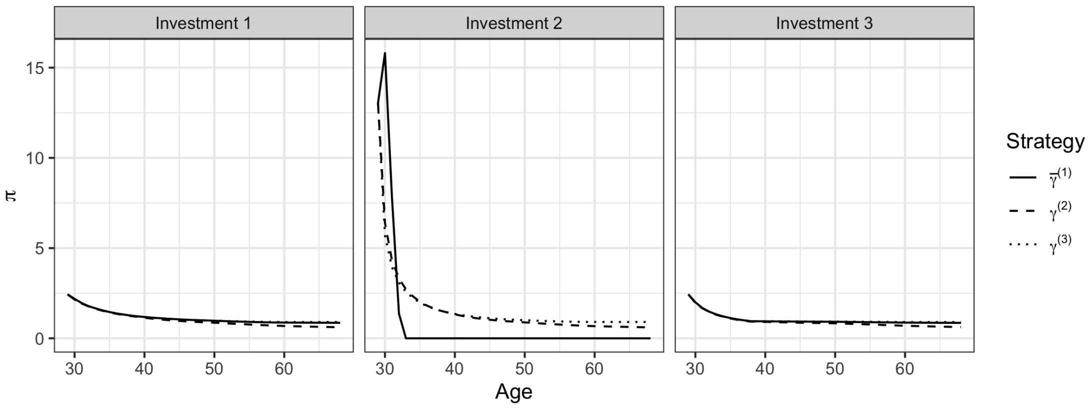

Figure 5.8: *Glide paths for the moderate risk profile in pension with the three investments from the real estate category where π is the weight in asset class 5–10.*

In Section 4.1, we concluded that estimating risk aversion in the gearing phase of a pension product would cause too high-risk aversions. When combining the constrained pension allocation with the geared position in real estate, we achieve a glide path that seems more optimal without projecting the savings account. The risk aversions in Figure 5.9 are within a reasonable level for the whole life-cycle. However, they have three plateaus, depending on whether the short, middle or long-term physical market parameters are used. It is clear from all investments in Figure 5.9 that we achieve the most consistent risk aversion with $\bar{\gamma}^{(1)}$ and the best alternative to decrease computational time is $\gamma^{(3)}$. As the second projection strategy makes the savings account develop faster than the free capital, the risk aversion increases over time. The second real estate investment has the lowest risk aversion, and the third has the highest, which seems successful.

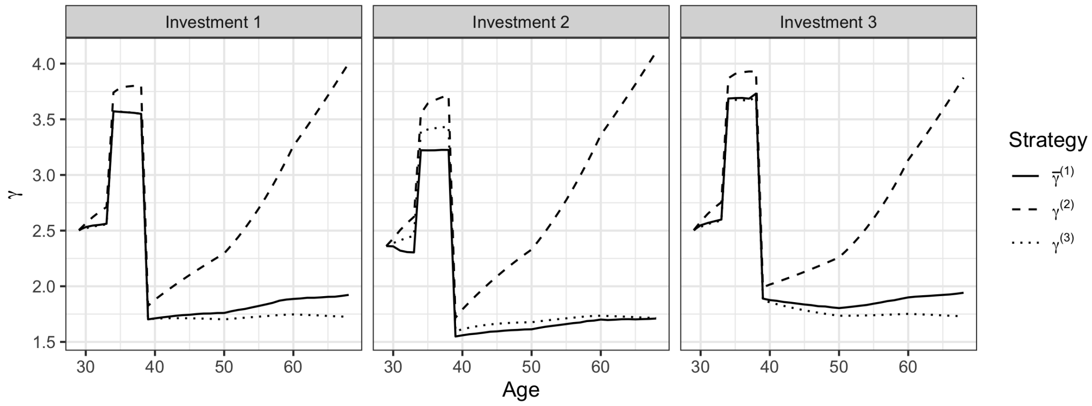

Figure 5.9: *Risk aversion, y-axis, for the moderate risk profile in pension with the three investments from real estate.*

We turn to calculate the CEB upon retirement from Section 4.4, where we use the $H = 10,000$ simulations from the Monte Carlo simulation. In the following, we use the mean risk aversions from Figure 5.9 given by

$$\hat{\gamma}^{(1)} = \frac{1}{T}\sum_{t=0}^{T}\bar{\gamma}^{(1)}(t),$$

to account for the changing risk aversion over time. Then $\hat{\gamma}^{(1)}$ is input along with the simulated wealth processes in the formula for CEB upon retirement at age 65, Equation (4.17). We consider again the 28-year-old investor with the moderate risk profile and the first real estate investment from Table 5.4 who receives an amount of money. These are distributed to the savings account or the free capital. The distribution is done yearly or as a lump sum in the first year. All installments are calculated with a life annuity to fit the present value of the lump sum. For the moderate risk profile and the first real estate investment, we study CEB when distributing 500,000 DKK in the following ways: 1) Shorten the mortgage with 11 years by paying instead 103,392 DKK every year, 2) pay 32,392 DKK extra to the savings account for the next 20 years, 3) insert the 500,000 DKK into the savings account today or 4) invest the 500,000 DKK in the real estate such that the free capital increases to 1,500,000 DKK and the mortgage stays the same.

$\hat{\gamma}^{(1)} = 2.23$							
Investment	t_ψ	l_ψ	ψ_0	$X_{\text{Real estate}}$	l_{Pension}	X_{Pension}	CEB
Original	30	0.071	1.5	1.0	0.045677	0.035839	18.62
1	19	0.103392	1.5	1.0	0.045677	0.035839	18.65
2	30	0.071	1.5	1.0	$0.045677 + 1_{t\leq 20}0.032392$	0.035839	19.62
3	30	0.071	1.5	1.0	0.045677	0.535839	20.35
4	30	0.071	1.5	1.5	0.045677	0.035839	20.42

Table 5.5: *Investment strategies for the distribution of 500,000 DKK for the moderate risk profile with the first real estate investment. Here t_ψ is mortgage length in years, l_ψ is yearly mortgage installments, ψ_0 is the mortgage today, $X_{Real\ estate}$ is the free capital today, $l_{Pension}$ is the yearly premium and $X_{pension}$ is the savings account today all stated in millions.*

Table 5.5 shows that investment four, where the investor increases her real estate investment, leads to the highest CEB upon retirement. However, the additional pension payments lead to the highest CEB when distributing the lump sum yearly instead. These observations align with the riskiest investment being advantageous for a younger individual. We have not considered if there emerges any change in risk aversion after the lump sum distribution, i.e., how sensitive the estimated risk aversion is to such a financial action.

The CEB metric is highly dependent on having estimated the correct risk aversion. From the CEB metric, a way of communicating the risk aversion to the investor emerges. We can present the CEB upon retirement to the investor as a dialogue starting point. We can further calibrate the risk aversion by getting the investor's opinion. We also mentioned in Section 4.5 that an analytical approach uses the optimal value function. The study of CEB upon retirement leads to providing financial advice on relevant investment strategies.

6 Conclusion

Throughout this thesis, we have studied the risk aversion coming from optimal investment. We began our studies by establishing the Hamilton-Jacobi-Bellman equation using stochastic control theory. This equation led to deriving the optimal choices for a pension-saver with an uncertain lifetime. As risk-free premium or mortgage payments to the wealth process needed to be replicable in the market for a solution to exist, we added a risk-free asset. The theory containing a risk-free asset was presented in one dimension using Merton's mutual funds theorem. For the rest of the thesis, we studied how much weight the investor should invest in a risky fund containing the market's assets. We defined a life-cycle pension product from the optimal investment result, which we characterized by a gearing, de-escalation, and retirement phase.

We suggested two new approaches to estimate risk aversions by assuming the real asset allocation to be optimal. The first approach was based on solving for risk aversion in an overdetermined system of equations in a market with ten asset classes. With this approach, we estimated negative, hence undefined, risk aversions and quantities far above previous literature. We concluded the method was too sensitive to the interest rate, correlations in the market, and assuming a constrained optimal strategy to be optimal.

In the second approach, we assumed that the real asset allocation was a risky fund as defined in Merton's mutual funds theorem. We redefined the market only to contain one fund and used the risky fund's market parameters to estimate the risk aversion. From this point, we could take any portfolio from pension, real estate, free funds, or a combination of the three and define a risky fund in which the investor had everything. The risk aversions estimated with this method were more robust to the interest rate and gave values aligned with the literature.

The estimation of a pension-savers risk aversion went smoothly in the de-escalation and retirement phase. However, we could not estimate a meaningful risk aversion for a pension-saver in the gearing phase where the investment strategy was constrained optimal. We argued that real life-cycle products followed an age-dependent version of the optimal investment strategy. Therefore we assessed the risk aversion when the pension-saver exited the gearing phase by considering three projection strategies to make the stochastic savings account age-dependent: Monte Carlo simulation, approximation with a geometric Brownian motion, and accumulating with the risk-free returns and mortality rates only. We concluded that the time-consuming Monte Carlo simulation was not much better than the latter strategy. As we could now estimate risk aversion forward in time, we interpreted variability in risk aversion as a sign of the sub-optimality of the real investment strategy. We tested the theory on data from a pension company with three risk profiles and found consistent risk aversions over risk profiles and age groups.

Using the framework on real estate investments and possibly free funds investments, we wanted to give a general view of the investor's intentional or unintentional financial risks. The risk aversions in real estate investments were lower than in the pension category, which matches the risky nature of shorting bonds to gear real estate. The real estate investment followed the nature of a life-cycle pension product as the geared position was reduced when paying off the mortgage, creating a glide path. A limitation for low initial values of the free capital was that when simulating the free capital process, it often became zero and could not recover.

Lastly, we found a mutual risk aversion by considering an investor's total wealth. We constructed a total wealth process where the savings account acted as a buffer for the free capital, reducing the problem of negative free capital. We completed the mutual category by using the projection machinery to estimate certainty equivalent balances upon retirement. If a young investor with a moderate pension and a standard mortgage was bestowed with a lump sum, it maximized the certainty equivalent balance at retirement if the money were invested as risky as possible.

We can now estimate and compare risk aversions coming from different life-cycle pension products, real estate investments, and in principle also, investments of free funds in the market. Together we can estimate a mutual risk aversion that can be analyzed across different groups of investors. Furthermore, we can advise investors if the market changes or how to allocate income in a way that fits their risk aversion and optimizes their utility.

6.1 Future research

In the pension category, the search for risk aversions has been limited to life-cycle pension products without any guarantees. There are, however, many products not under this description – sometimes, there are even guarantees in life-cycle products. Further studies might shed light on estimating an investor's risk aversion in the broader spectrum of insurance products. Examples could be to study risk aversions coming from other optimal choices such as benefit schemes or optimal life insurance. Stochastic premium creates another optimal investment strategy that might identify other pension products to be studied.

A sparse amount of attention has been given to modeling the real estate market. We concluded that modeling the free capital as a positive process was unrealistic and problematic, especially for young investors. This limitation could be solved by studying other markets than the Black-Scholes market and other utility functions than the power utility function. Furthermore, in real estate, we have only considered one set of market parameters to describe all real estate investments and a life annuity to describe all mortgages. The spectrum of loan types and coupon rates is left to be studied.

Communicating the results through certainty equivalents can give a new foundation of comparisons between investors that we have yet to explore. It might be relevant to study further implementations of certainty equivalents or perform questionnaires to calibrate the estimated risk aversion through certainty equivalents as in [Burgaard and Steffensen, 2020].

In this thesis, we have not implemented taxes or expenses of any kind. In Denmark, an investor can postpone paying taxes by investing in a pension product and, in that way, avoid paying extra taxes due to a high income. Instead, a lower tax is paid on the capital gains on the savings account. These effects significantly impact the optimal choices and could change the conclusion in our example with certainty equivalent balance upon retirement.

A Appendix

A.1 Proof of Theorem 4.1

Proof. From page 355–358 in [Bjørk, 2020]

$$\tilde{\boldsymbol{w}} = \underbrace{\frac{1}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}_{\boldsymbol{A}} + \frac{V_x}{xV_{xx}}\underbrace{\boldsymbol{\Sigma}^{-1}\left(\frac{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{e} - \boldsymbol{\alpha}\right)}_{\boldsymbol{B}},$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times 1$-dimensional vectors.

By differentiation, the first-order constraint on consumption is given by

$$F_c(t,c) = V_x \iff \tilde{\boldsymbol{c}} = e^{-\frac{1}{\gamma}\rho t}V_x^{-\frac{1}{\gamma}}.$$

Second, we guess the following ansatz for $V(t,x)$ with its partial derivatives

$$\begin{aligned} G^{\tilde{w}}(t,x) &= \frac{1}{1-\gamma}g(t)x^{1-\gamma} \quad , \quad G_t^{\tilde{w}}(t,x) = \frac{1}{1-\gamma}g_t(t)x^{1-\gamma}, \\ G_x^{\tilde{w}}(t,x) &= g(t)x^{-\gamma} \quad , \quad G_{xx}^{\tilde{w}}(t,x) = -\gamma g(t)x^{-\gamma-1}, \end{aligned}$$

with limit condition $g(N) = 0$. The weight in the risky assets is given by

$$\begin{aligned} \tilde{\boldsymbol{w}} &= \frac{1}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{\Sigma}^{-1}\boldsymbol{e} - \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\frac{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\boldsymbol{e}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{e}}\boldsymbol{e} - \boldsymbol{\alpha}\right) \\ &= \boldsymbol{A} - \frac{1}{\gamma}\boldsymbol{B}. \end{aligned}$$

The optimal consumption is given by

$$\tilde{\boldsymbol{c}} = xe^{-\frac{1}{\gamma}\rho t}g(t)^{-\frac{1}{\gamma}}.$$

By inserting $\tilde{\boldsymbol{w}}$, $\tilde{\boldsymbol{c}}$ and the derivatives of the ansatz in the HJB equation, the following equation for g needs to be satisfied for the HJB equation to hold

$$\begin{aligned} 0 &= \frac{1}{1-\gamma}g_t(t)x^{1-\gamma} + \frac{1}{1-\gamma}e^{-\rho t}e^{-\frac{1-\gamma}{\gamma}\rho t}x^{1-\gamma}g(t)^{-\frac{1-\gamma}{\gamma}} + \left(x\left(\boldsymbol{A}^T - \frac{1}{\gamma}\boldsymbol{B}^T\right)\boldsymbol{\alpha}g(t)x^{-\gamma}\right) \\ &\quad - e^{-\frac{1}{\gamma}\rho t}g(t)^{-\frac{1-\gamma}{\gamma}}x^{1-\gamma} - x^2\frac{1}{2}\left(\boldsymbol{A}^T - \frac{1}{\gamma}\boldsymbol{B}^T\right)\boldsymbol{\Sigma}\left(\boldsymbol{A} - \frac{1}{\gamma}\boldsymbol{B}\right)\gamma g(t)x^{-\gamma-1} \\ &= g_t(t) + g(t)(1-\gamma)\underbrace{\left(\boldsymbol{A}^T - \frac{1}{\gamma}\boldsymbol{B}^T\right)\left(\boldsymbol{\alpha} - \frac{1}{2}\boldsymbol{\Sigma}\left(\boldsymbol{A} - \frac{1}{\gamma}\boldsymbol{B}\right)\right)}_{C} + e^{-\frac{1}{\gamma}\rho t}\gamma g(t)^{-\frac{1-\gamma}{\gamma}}, \end{aligned} \tag{A.1}$$

where $g(N) = 0$. This is known as a Bernoulli equation and is solved by substituting $u(t) = g(t)^{\frac{1}{\gamma}}$ to get

$$u_t(t) + \frac{1-\gamma}{\gamma} C u(t) = -e^{-\frac{1}{\gamma}\rho t}.$$

Solving this ODE, we get

$$u(t) = -\frac{\gamma}{(1-\gamma)C - \rho} + e^{-\frac{1}{\gamma}t\rho} + e^{-\frac{1}{\gamma}t(1-\gamma)C} k,$$

where k is the constant stemming from taking the antiderivative. By using the terminal condition $g(N) = 0 \Rightarrow g(N)^\gamma = u(N) = 0$, we get that

$$k = \frac{\gamma}{(1-\gamma)C - \rho} e^{\frac{1}{\gamma}N((1-\gamma)C-\rho)}.$$

Converting to $g(t)$ by using the inverse $g(t) = u(t)^\gamma$, we get

$$g(t) = e^{-\rho t}\left(\frac{\gamma}{(1-\gamma)C - \rho}\left(e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)} - 1\right)\right)^\gamma. \tag{A.2}$$

Plug into the optimal consumption, we get

$$\tilde{\boldsymbol{c}} = x e^{-\frac{1}{\gamma}\rho t} g(t)^{-\frac{1}{\gamma}} = x\left(\frac{\gamma}{(1-\gamma)C - \rho}\left(e^{\frac{1}{\gamma}((1-\gamma)C-\rho)(N-t)} - 1\right)\right)^{-1}.$$

For the special case $\gamma = 1$, we guess the ansatz $G(t, x) = g(t)\log(x) + h(t)$ such that we only need to change the HJB equation in Equation A.1 to

$$0 = g_t(t)\left(\log(x) + e^{-\rho t}\right) + h_t(t) - e^{-\rho t}\left(\rho t + \log(g(t)) + 1\right) - g(t)C.$$

We now get the same g as in Equation A.2 and

$$h(t) = -\int_t^T e^{-\rho s}\left(\rho s + \log(g(s)) + 1\right) + g(s)C ds.$$

Therefore, g, $\boldsymbol{A}$, $\boldsymbol{B}$, C, $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{c}}$ remains the same. □

A.2 Time-inconsistent theory

Definition A.1. *(Definition 2 in [Kryger et al., 2020])*
Consider an arbitrary strategy π. If there exists functions $Y^\pi(t, x), Z^\pi(t, x) \in C^{1,2}$ such that

$$\begin{aligned} Y^\pi_t &= -(r + \pi(\alpha - r))xY^\pi_x - \frac{1}{2}\sigma^2\pi^2x^2Y^\pi_{xx}, \\ Y^\pi(T, x) &= g(x), \end{aligned}$$

and

$$\begin{aligned} Z^\pi_t &= -(r + \pi(\alpha - r))xZ^\pi_x - \frac{1}{2}\sigma^2\pi^2x^2Z^\pi_{xx}, \\ Z^\pi(T, x) &= h(x), \end{aligned}$$

and such that the processes

$$\sigma\pi(s)X^\pi(s)Y_x^\pi\left(s, X^\pi(s)\right),$$
$$\sigma\pi(s)X^\pi(s)Z_x^\pi\left(s, X^\pi(s)\right),$$

are in $\mathcal{L}^2$ *and such that the function* $\bar{f}(t,x) := f\left(t, x, Y^\pi(t,x), Z^\pi(t,x)\right)$ *is in* $C^{1,2}$*, then the strategy* π *is called admissible w.r.t. the functions* $Y^\pi(t,x)$ *and* $Z^\pi(t,x)$*.*

Theorem A.2. *(Theorem 1 in [Kryger et al., 2020])*
Let $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function. Let* g *and* h *be real functions. The set of admissible strategies is given by Definition 2. Note that admissibility depends on the choice of* g, h*. Consider the investor with value function*

$$J^\pi(t,x) := f\left(t, x, y^\pi(t,x), z^\pi(t,x)\right),$$

with

$$y^\pi(t,x) = E_{t,x}\left[g\left(X^\pi(T)\right)\right],$$
$$z^\pi(t,x) = E_{t,x}\left[h\left(X^\pi(T)\right)\right].$$

Denote by $V(t,x)$ *the optimal value function in the sense of an equilibrium control law of Definition 1. If there exist two functions* $G, H \in C^{1,2}$ *such that the control law*

$$\tilde{\pi} = \arg\inf_\pi \left[-(r + \pi(\alpha - r))x\left(f_y G_x + f_z H_x\right) - \frac{1}{2}\sigma^2\pi^2x^2\left(f_y G_{xx} + f_z H_{xx}\right)\right],$$

is an admissible strategy w.r.t. G*, and* H*, then*

$$y^{\tilde{\pi}}(t,x) = G(t,x), z^{\tilde{\pi}}(t,x) = H(t,x),$$

and the optimal investment strategy is given by $\tilde{\pi}$*, and the optimal value function is determined by*

$$V(t,x) = f(t, x, G(t,x), H(t,x)).$$

A.3 Derivation of Example 4.13

The value function is given by

$$J^\pi(t,x) := f\left(t, x, y^\pi(t,x), z^\pi(t,x)\right),$$

where

$$f(t,x,y,z) = u^{-1}(y^\pi(t,x)) + u^{-1}(z^\pi(t,x))$$
$$= y^\pi(t,x)^{\frac{1}{1-\gamma_1}} + z^\pi(t,x)^{\frac{1}{1-\gamma_1}},$$

with

$$y^\pi(t,x) = E_{t,x}\left[g\left(X^\pi(T)\right)\right], \quad z^\pi(t,x) = E_{t,x}\left[h\left(X^\pi(T)\right)\right].$$

The value function has derivatives

$$f_y = \frac{1}{1-\gamma_1} y^{\frac{\gamma_1}{1-\gamma_1}} = \frac{1}{1-\gamma_1} x^{\gamma_1} b_1(t)^{\gamma_1}, \qquad f_z = \frac{1}{1-\gamma_2} z^{\frac{\gamma_2}{1-\gamma_2}} = \frac{1}{1-\gamma_2} x^{\gamma_2} b_2(t)^{\gamma_2}.$$

We guess the solutions

$$y^{\tilde{\pi}}(t,x) = G(t,x) = x^{1-\gamma_1} b_1(t)^{1-\gamma_1}, \quad z^{\tilde{\pi}}(t,x) = H(t,x) = x^{1-\gamma_2} b_2(t)^{1-\gamma_2}.$$

$G(t,x)$ and $H(t,x)$ are assumed to satisfy the admissibility condition in Definition A.1, i.e., they solve a Feynman-Kač differential equation with terminal condition

$$G(x,T) = g(x) \Longleftrightarrow x^{1-\gamma_1} b_1(T)^{1-\gamma_1} = \left(\frac{x}{n}\right)^{1-\gamma_1} \Longleftrightarrow b_1(T) = \frac{1}{n}.$$

The same calculations reveal that $b_2(T) = \frac{1}{n}$. The derivatives are given by

$$G_x = (1-\gamma_1)\, x^{-\gamma_1} b_1(t)^{1-\gamma_1}, \qquad G_{xx} = -\gamma_1\,(1-\gamma_1)\, x^{-\gamma_1-1} b_1(t)^{1-\gamma},$$

and

$$H_x = (1-\gamma_2)\, x^{-\gamma_2} b_2(t)^{1-\gamma_2}, \qquad G_{xx} = -\gamma_2\,(1-\gamma_2)\, x^{-\gamma_2-1} b_2(t)^{1-\gamma_2}.$$

In order to find $\tilde{\pi}$, the weight in the risky asset, note that

$$f_y G_x = b_1(t), \quad f_y G_{xx} = -\gamma_1 x^{-1} b_1(t), \quad f_z H_x = b_2(t), \qquad f_z H_{xx} = -\gamma_2 x^{-1} b_2(t).$$

Then

$$\begin{aligned}
\tilde{\pi} &= \underset{\pi}{\operatorname{arginf}} \left[-(r + \pi(\alpha - r))x\,(f_y G_x + f_z H_x) - \frac{1}{2}\sigma^2 \pi^2 x^2\,(f_y G_{xx} + f_z H_{xx}) \right] \\
&= \underset{\pi}{\operatorname{arginf}} \Bigg[-(r + \pi(\alpha - r))x\,(b_1(t) + b_2(t)) - \frac{1}{2}\sigma^2 \pi^2 x\,(-\gamma_1 b_1(t) - \gamma_2 b_2(t)) \Bigg].
\end{aligned}$$

Finding the first order condition and isolating for π, the optimal weight in the risky asset is given by

$$\tilde{\pi} = \frac{(\alpha - r)\,(b_1(t) + b_2(t))}{\sigma^2\,(\gamma_1 b_1(t) + \gamma_2 b_2(t))}.$$

Insert in Feynman-Kač representation of G from Definition A.1 to get

$$\begin{aligned}
0 &= -x^{1-\gamma_1}(1-\gamma_1) b_1(t)^{-\gamma_1} b_1'(t) - \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))} x^{1-\gamma_1}(1-\gamma_1) b_1(t)^{1-\gamma_1} \\
&\quad - r x^{1-\gamma_1}(1-\gamma_1) b_1(t)^{1-\gamma_1} + \frac{1}{2} \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))^2}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))^2} \gamma_1 x^{1-\gamma_1}(1-\gamma_1) b_1(t)^{1-\gamma_1} \\
&= -b_1'(t) - \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))} b_1(t) - r b_1(t) + \frac{1}{2} \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))^2}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))^2} \gamma_1 b_1(t).
\end{aligned}$$

The same calculation for H reveals a system of ODE's

$$0 = -b_2'(t) - \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))} b_2(t) - r b_2(t) + \frac{1}{2} \frac{(\alpha - r)^2\,(b_1(t) + b_2(t))^2}{\sigma^2\,(\gamma_1 b_1(t) \gamma_2 b_2(t))^2} \gamma_2 b_2(t).$$

A.4 The market proposed by "Rådet for Afkastforventninger"

The volatility vectors for year $[1, 11)$ are given by

$2022Q3-4$		Volatility
1	Government- and mortgage bonds	3.3%
2	Investment-grade bonds	4.4%
3	High-yield bonds	8.9%
4	Emerging markets government bonds	9.9%
5	Global stocks (developed markets)	15.4%
6	Emerging markets stocks	21.9%
7	Private equity	20.2%
8	Infrastructure	13.5%
9	Real estate	10.8%
10	Hedgefunds	9.1%

, (A.3)

$2023Q1-2$		Volatility
1	Government- and mortgage bonds	3.7%
2	Investment-grade bonds	5.5%
3	High-yield bonds	11.9%
4	Emerging markets government bonds	10.7%
5	Global stocks (developed markets)	15.3%
6	Emerging markets stocks	21.7%
7	Private equity	20.4%
8	Infrastructure	14.0%
9	Real estate	10.8%
10	Hedgefunds	9.4%

.

The correlation matrices for the years [1,11) with the same asset classes as in Equation (A.3) are given by

$2022Q3-4$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1.0	0.6	0.1	0.4	−0.1	−0.1	−0.3	−0.1	−0.2	−0.1
2	0.6	1.0	0.5	0.5	0.2	0.2	0.1	0.1	0.1	0.3
3	0.1	0.5	1.0	0.7	0.7	0.6	0.6	0.4	0.3	0.7
4	0.4	0.5	0.7	1.0	0.4	0.6	0.4	0.2	0.2	0.5
5	−0.1	0.2	0.7	0.4	1.0	0.7	0.8	0.4	0.4	0.8
6	−0.1	0.2	0.6	0.6	0.7	1.0	0.7	0.3	0.4	0.8
7	−0.3	0.1	0.6	0.4	0.8	0.7	1.0	0.4	0.5	0.8
8	−0.1	0.1	0.4	0.2	0.4	0.3	0.4	1.0	0.4	0.4
9	−0.2	0.1	0.3	0.2	0.4	0.4	0.5	0.4	1.0	0.4
10	−0.1	0.3	0.7	0.5	0.8	0.8	0.8	0.4	0.4	1.0

,

$2023Q1-2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	0.6	0.1	0.3	−0.1	−0.1	−0.2	−0.1	−0.1	−0.1
2	0.6	1	0.6	0.6	0.2	0.2	0.2	0.1	0.1	0.3
3	0.1	0.6	1	0.7	0.7	0.6	0.6	0.4	0.3	0.7
4	0.3	0.6	0.7	1	0.5	0.6	0.4	0.2	0.2	0.5
5	−0.1	0.2	0.7	0.5	1	0.7	0.8	0.4	0.4	0.8
6	−0.1	0.2	0.6	0.6	0.7	1	0.7	0.4	0.4	0.7
7	−0.2	0.2	0.6	0.4	0.8	0.7	1	0.4	0.4	0.7
8	−0.1	0.1	0.4	0.2	0.4	0.4	0.4	1	0.3	0.4
9	−0.1	0.1	0.3	0.2	0.4	0.4	0.4	0.3	1	0.4
10	−0.1	0.3	0.7	0.5	0.8	0.7	0.7	0.4	0.4	1

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The expected returns for year $[1, 6)$ are given by

$2022Q3-4$		Expected return
1	Government- and mortgage bonds	0.0%
2	Investment-grade bonds	0.3%
3	High-yield bonds	2.0%
4	Emerging markets government bonds	2.7%
5	Global stocks (developed markets)	5.0%
6	Emerging markets stocks	7.6%
7	Private equity	8.4%
8	Infrastructure	4.1%
9	Real estate	2.5%
10	Hedgefunds	2.2%

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$2023Q1-2$		Expected return
1	Government- and mortgage bonds	1.9%
2	Investment-grade bonds	2.2%
3	High-yield bonds	4.9%
4	Emerging markets government bonds	4.3%
5	Global stocks (developed markets)	6.1%
6	Emerging markets stocks	8.3%
7	Private equity	10.2%
8	Infrastructure	5.6%
9	Real estate	4.1%
10	Hedgefunds	3.8%

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The expected returns for year $[6, 11)$ are given by

$2022Q3-4$		Expected return
1	Government- and mortgage bonds	1.0%
2	Investment-grade bonds	1.6%
3	High-yield bonds	3.6%
4	Emerging markets government bonds	3.7%
5	Global stocks (developed markets)	5.9%
6	Emerging markets stocks	8.7%
7	Private equity	9.5%
8	Infrastructure	5.3%
9	Real estate	4.7%
10	Hedgefunds	3.5%

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$2023Q1-2$		Expected return
1	Government- and mortgage bonds	2.1%
2	Investment-grade bonds	3.1%
3	High-yield bonds	5.0%
4	Emerging markets government bonds	4.7%
5	Global stocks (developed markets)	6.3%
6	Emerging markets stocks	8.8%
7	Private equity	10.5%
8	Infrastructure	5.9%
9	Real estate	5.1%
10	Hedgefunds	4.3%

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After 11 years, the correlation between stocks, illiquid assets, and bonds is zero. Therefore the expected returns and volatilities are given by

$2022Q3-4$	Expected return	Volatility
Bonds	3.5%	8.0%
Stocks	6.5%	18.0%

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$2023Q1-2$	Expected return	Volatility
Bonds	3.5%	8.0%
Illiquid assets	6.5%	12.0%
Stocks	6.5%	18.0%

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