國立交通大學 105 學年度碩士班考試入學試題

科目：線性代數與離散數學(1002)　　考試日期：105 年 2 月 3 日 第 2 節

系所班別：資訊聯招　　

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Determine and explain whether each of the following sets is a subspace of $\mathbf{R}^{2x2}$.
   a. **(2 points)** The set of all 2x2 triangular matrices.
   b. **(2 points)** The set of all 2x2 lower triangular matrices.
   c. **(2 points)** The set of all 2x2 orthogonal matrices.

2. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be linearly independent (LID) vectors in $\mathbf{R}^n$. Determine and explain whether each of the following vector sets is LID.
   a. **(2 points)** $\mathbf{y}_1=\mathbf{x}_1+\mathbf{x}_2$, $\mathbf{y}_2=\mathbf{x}_2+\mathbf{x}_3$, and $\mathbf{y}_3=\mathbf{x}_3+\mathbf{x}_1$.
   b. **(2 points)** $\mathbf{z}_1=\mathbf{x}_2-\mathbf{x}_1$, $\mathbf{z}_2=\mathbf{x}_3-\mathbf{x}_2$, and $\mathbf{z}_3=\mathbf{x}_3-\mathbf{x}_1$.

3. **(3 points)** Let $\mathbf{v}_1=(2,6)^T$, $\mathbf{v}_2=(1,4)^T$, and $\mathbf{S}=\begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}$. Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that $\mathbf{S}$ will be the transition matrix from the ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

4. **(6 points)** Please find a basis for the column space and a basis for the null space of $\mathbf{A}=\begin{bmatrix}1 & 3 & -2 & 1\\ 2 & 1 & 3 & 2\\ 3 & 4 & 5 & 6\end{bmatrix}$.

5. **(6 points)** Three persons, TC, CC, and YJ, have their own gold. They always divide their own gold into two equal parts and donate to others every day. TC, for example, has 12-kilograms gold today and donate 6 kilograms to CC and the other 6 kilograms to YJ. If TC has 6 kilograms, CC has 1 kilogram, and YJ has 2 kilograms at first, how many kilograms of gold do CC and YJ have after 365 days?

6. Answer true (T) or false (F) to each of the following statements.
   a. **(1 point)** The linear transformation $T(f)=f'$ from P_n to P_n has rank *n* for all positive integers *n*.
   b. **(1 point)** For an $m \times n$ matrix $\mathbf{A}$, $m > n$, the nullity of $\mathbf{A}^T$ must be larger than the nullity of $\mathbf{A}$.
   c. **(1 point)** If U, V, and W are subspaces of $\mathbf{R}^3$, $U \perp V$, and $V \perp W$, then $U \perp W$.
   d. **(1 point)** For any matrix $\mathbf{A}$, the eigenvalues of $\mathbf{A}^T\mathbf{A}$ are always nonnegative.
   e. **(1 point)** There exist orthogonal 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}+\mathbf{B}$ is orthogonal as well.
   f. **(1 point)** If $\mathbf{A}\mathbf{A}^T=\mathbf{A}^2$ for a 2×2 matrix $\mathbf{A}$, then $\mathbf{A}$ must be symmetric.
   g. **(1 point)** If the eigenvalues of a matrix are not distinct, then this matrix is not diagonalizable.
   h. **(1 point)** For a Hermitian matrix, its eigenvectors belonging to distinct eigenvalues are not only linearly independent but also orthogonal.
   i. **(1 point)** An orthogonally diagonalizable matrix is always symmetric.
   j. **(1 point)** Any two matrices with the same trace are similar.

7. Consider a matrix $\mathbf{A}=\begin{bmatrix}a & b\\ 0 & c\end{bmatrix}$, $a \neq c$.
   a. **(6 points)** Factor the matrix $\mathbf{A}$ into $\mathbf{XDX}^{-1}$ where $\mathbf{D}$ is diagonal.
   b. **(4 points)** Calculate $e^{\mathbf{A}}$.

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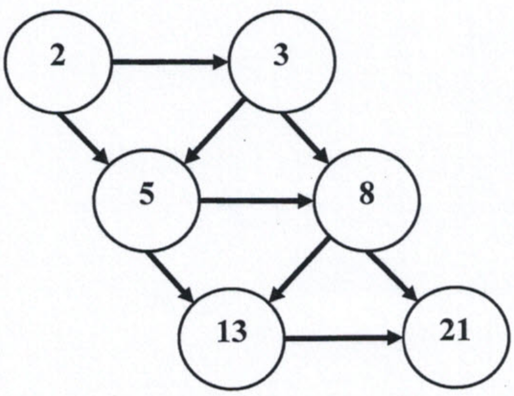
8. **(5 points)** Prove that the inverse of a symmetric invertible matrix is also symmetric.

9. In the following questions, just give the answer, do not give any explanation.
   a. **(3 points)** Determine whether the following proposition is a tautology:
   $$((p \to \neg q) \land q) \to \neg p$$
   b. **(3 points)** Suppose $g : A \to B$ and $f : B \to C$, where $f \circ g$ is 1-1. Determine which of the following statements is correct:
   (1) if g is 1-1, then f is 1-1;
   (2) if f is 1-1, then g is 1-1;
   (3) both of (1) and (2) are correct;
   (4) both of (1) and (2) are incorrect.
   c. **(3 points)** Suppose $f: \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ where $g(x) = 2x + 1$ and $g \circ f(x) = 2x + 11$. Find the rule for f.

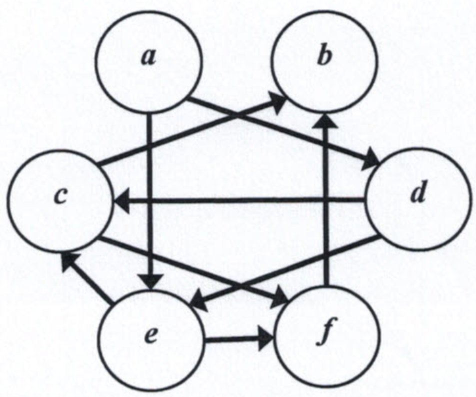
10. In the following questions, just give the answer, do not give any explanation.
    a. **(5 points)** Find $11^{121} \bmod 13$.
    b. **(5 points)** Find all solutions of the system of congruences $x \equiv 4 \pmod 6$ and $x \equiv 13 \pmod{15}$.
    c. **(5 points)** Determine a rule for generating the terms of the sequence that begins 1, 3, 4, 8, 15, 27, 50, 92,…, and find the next three terms of the sequence.

11. **(5 points)** Please use a combinatorial argument to prove the equality
$$\binom{n}{r} = \sum_{j=r}^{n} \binom{j-1}{r-1}$$
by considering the r-combination of the set $\{1,2,\cdots,n\}$.

12. Please answer the following questions based on the two graphs below.



(I)



(II)

Graph (I) is part of the relation in the Fibonacci sequence. Let R denote the relation represented by Graph (I).
   a. **(3 points)** Are Graph (I) and Graph (II) isomorphic? If your answer is NO, please give your reasons; otherwise, please show the correspondence between the two vertex sets.
   b. **(1 points)** Is Graph (II) planar?
   c. **(5 points)** Please give the 0-1 matrix of the composite relation R^4 under the vertex ordering 2,3,5,8,13,21. What is the meaning of the $(1,6)$ entry in the matrix.

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13. Solve the linear recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 6n - 8$ with $a_0 = 5$ and $a_1 = 10$ by the following steps.

a. **(2 points)** Find the characteristic equation of the associated homogeneous recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

b. **(3 points)** Find the solutions $a_n^{(h)}$ of the associated homogeneous recurrence relation.

c. **(5 points)** Find a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 6n - 8$.

d. **(2 points)** Use the initial condition $a_0 = 5$ and $a_1 = 10$ to find the solution.