【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

First you need to write down your answers clearly and then explain how to compute the answers. You also need to answer the questions in order. Do not jump around.

1. (a) (10 points) Please translate the following two assertions into logical formulae. We consider natural numbers only. Assume that there is a predicate $IsPrime(x)$, which is true if and only if x is a prime number. Hence you do not need to define prime numbers. An *interval*, denoted as $[a, b]$, is defined as the set $\{n | n \in N, a \leq n \leq b\}$. The size of this interval is $b - a + 1$.

- There is a prime number in every interval whose size is 10^8.
- There are infinite pairs of consecutive prime numbers (p, q) satisfying $|p - q| \leq 10^8$.

(b) (5 points) Are the above two assertions logically equivalent? Which must be false? Why?

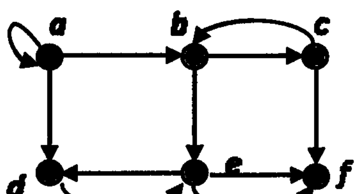
2. (10 points) Assume $n = 2^k$, $k \in N$. The straightforward way to multiply two $n \times n$ matrices takes time $O(n^3)$.

(a) (5 points) Please describe an asymptotically faster multiplication method.

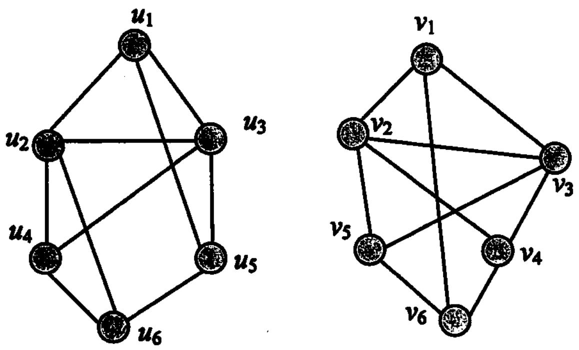
(b) (5 points) Use the recurrence relation to analyze the time complexity of your method.

3. Assume that each person i is asked to arbitrarily select n_i different integer(s) from the range $\{1, 2, \ldots, r_i\}$.
(a) (5 points) If $r_u = r_v = k$ for two persons u and v, what is the maximal value of k (expressed in terms of n_u, n_v and/or other constants if needed) that guarantees at least one common integer selected by both u and v?
(b) (5 points) If $r_u = 5$, $n_u = 3$, $r_v = 7$, and $n_v = 3$, what is the number of combinations of u's and v's choices in which u and v share exactly one common integer?

4. **(5 points)** Consider a relation $\angle$ represented by the following digraph, where a directed edge from node i to node j exists if and only if $i \angle j$. Consider another relation $\approx$ defined as $i \approx j$ if and only if neither $i \angle j$ nor $j \angle i$. If we represent $\approx$ using a zero-one matrix, how many 1s are there in the matrix?



5. **(5 points)** Determine whether the following two graphs are isomorphic or not. Exhibit an isomorphism (i.e., $u_1 = v_1$, $u_2 = v_4$, etc.) or provide a rigorous argument that none exists.



6. **(5 points)** Solve the following recurrence relation together with the initial condition given.
$a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 4$, $a_1 = 4$

7. Let **A** be a non-symmetric n by n matrix. Determine and explain whether the following matrix (**B**, **C**): ① must be symmetric, ② must be skew-symmetric, or ③ otherwise.
a. **(2 points)** $\mathbf{B} = \mathbf{A} - \mathbf{A}^T$
b. **(2 points)** $\mathbf{C} = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A}^T)$

8. a. **(5 points)** Suppose **A** is a 4 by 5 matrix and **B** is a 5 by 7 matrix and $\mathbf{AB} = \mathbf{0}$. If rank(**B**) = 2, what is the maximum possible rank(**A**)? and why?
b. **(5 points)** Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in $\mathbf{R}^n$. Suppose **A** is the sum of three matrices: $\mathbf{A} = \boldsymbol{u}\boldsymbol{v}^T + \boldsymbol{v}\boldsymbol{w}^T + \boldsymbol{w}\boldsymbol{u}^T$. If $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w} = \mathbf{0}$ has a nonzero solution (x_1, x_2, x_3), what is the maximum possible rank(**A**)? and why?

9. **(6 points)** Given a matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 3 \\ -1 & -2 & 2 & 3 \end{bmatrix},$$

we can split $\boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ into $\boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r \in C(\mathbf{A}^T)$ and $\boldsymbol{x}_n \in N(\mathbf{A})$. What are $\boldsymbol{x}_r$ and $\boldsymbol{x}_n$?

10. **(5 points)** Given 3 points (t, b): (-1, -7), (1, 7), and (2, 21), what is the sum of squared error $\|e\|^2$ if we fit the closest line $b = C + Dt$ by the least square approximation?

11. Answer true (T) or false (F) to each of the following statements.
a. **(1 point)** Let $L : \Re^n \to \Re^n$ be a linear operator. If $L(\mathbf{x}_1) = L(\mathbf{x}_2)$, then the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ must be equal.
b. **(1 point)** An orthogonal matrix must have an eigenvalue $\lambda = 1$.
c. **(1 point)** A Markov matrix must have an eigenvalue $\lambda = 1$.
d. **(1 point)** A projection matrix has only two eigenspaces and each is an orthogonal complement to the other.

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e. **(1 point)** Singular matrices are never diagonalizable.

f. **(1 point)** For a real symmetric matrix, all of its eigenvalues are always real and all of its eigenvectors corresponding to different eigenvalues are always mutually orthogonal.

g. **(1 point)** For a square matrix, the product of its pivots is always equal to the product of its eigenvalues.

h. **(1 point)** A symmetric matrix cannot be similar to a non-symmetric matrix.

i. **(1 point)** If $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ always share the same eigenvalues.

j. **(1 point)** Singular values of any matrix are all nonnegative real numbers.

12. **(5 points)** Let **A**, **B**, **C**, and **D** be $n \times n$ matrices with **A** invertible. Prove that $\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = (\det \mathbf{A}) \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})$.

13. **(10 points)** Political affiliations change from one generation to the next. We consider three parties, labeled D, K, and T. In one generation, membership in D goes 80% to D, 10% to K, and 10% to T; membership in K goes 20% to D, 80% to K, and none to T; and membership in T goes 50% to D, 10% to K, and 40% to T. What is the long-term steady state for the three parties (membership ratios among these three parties)?