

國立交通大學 109 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(1102)

考試日期：109 年 2 月 4 日 第 2 節

系所班別：資訊聯招

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【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

1. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Note that a partial ordering relation is reflexive, antisymmetric, and transitive. If not a partial ordering relation, explain why it is not.
 [3 points] (a) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
 [3 points] (b) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
 [3 points] (c) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

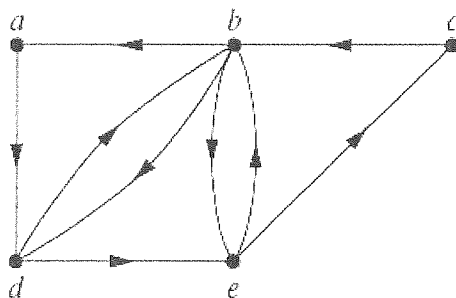
2. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \dots$ if

[4 points] (a) $a_n = 1 + (-1)^n$.

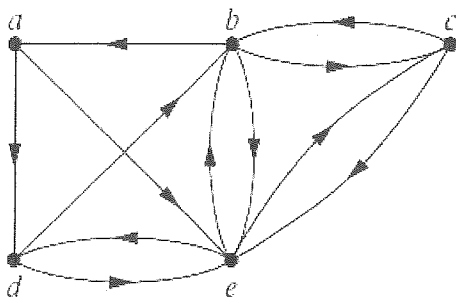
[4 points] (b) $a_n = n(n+1)$.

3. Determine whether the directed graph shown has an Euler circuit starting from vertex a . Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path also starting from vertex a . Construct a Euler path if one exists.

[4 points] (a)



[4 points] (b)



4. Consider a finite set N . Let $f: P(N) \rightarrow \mathbb{R}$ be a function from $P(N)$ to $\mathbb{R}$, where $P(N)$ is the power set of N and $\mathbb{R}$ is the set of all real numbers.

[4 points] (a) What is the maximum cardinality of the range of f ?

[4 points] (b) Suppose that for all $S_1, S_2 \subseteq N$ we have $f(S_1) + f(S_2) \leq f(S_1 \cup S_2) + f(S_1 \cap S_2)$.

Prove that for all finite sets C and T such that $C \subseteq T \subset N$ and $\forall i \in N - T$, we have $f(C \cup \{i\}) - f(C) \leq f(T \cup \{i\}) - f(T)$.

5. [5 points] (a) It is known that there are infinitely many prime numbers. We can prove this by contradiction as outlined below. Assume that $P = \{p_1, p_2, \dots, p_n\}$ is the set of all prime numbers, where $|P| = n$ and n is a finite number. Consider a number q . If q is a prime number, then we have a prime number not in P , a contradiction. If q is not a prime number, then none of

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$p_1, p_2, \dots, p_n$ divides q , which is another contradiction. We thus have the proof. What is the value of q to make the proof valid?

[5 points] (b) Let domain U be all positive integers in the range $[2, 10]$. Let propositional function $P(x)$ denote " x is a prime number," and $Q(x)$ denote " $x \equiv 3 \pmod{5}$." Find a set S with the maximum cardinality that makes " $\forall x \in S (P(x) \rightarrow Q(x))$ " true and " $\exists x \in S (P(x) \wedge Q(x))$ " false at the same time.

6. [7 points] If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa - tb$. Let $a = 277$ and $b = 91$ and find the values of s and t such that $\gcd(a, b) = sa - tb$.

7. Let E_1 , E_2 , and E_3 be 3×3 elementary matrices of type I (interchanging two rows of I), II (multiplying a row of I by a nonzero constant), and III (adding a multiple of one row to another row), respectively. Also let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from identity matrix by multiplying its second row by 3. Find the value of each of the following:

[1 point] (a) $\det(E_1 A)$

[1 point] (b) $\det(E_2 A)$

[1 point] (c) $\det(A E_3)$

[1 point] (d) $\det(E_1^2)$

[1 point] (e) $\det(E_3 E_2 E_1)$

8. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$,

[5 points] (a) Find elementary matrices F_1 , F_2 , and F_3 such that $F_3 F_2 F_1 A = U$, where U is an upper triangular matrix.

[5 points] (b) Determine the inverses of F_1 , F_2 , and F_3 and set $L = F_1^{-1} F_2^{-1} F_3^{-1}$. What type of matrix is L ? Verify that $A = LU$.

9. Let $A = [a_1 \ a_2 \ a_3 \ a_4]$ be a 4×4 matrix with reduced row echelon form given by $U =$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ If } a_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix},$$

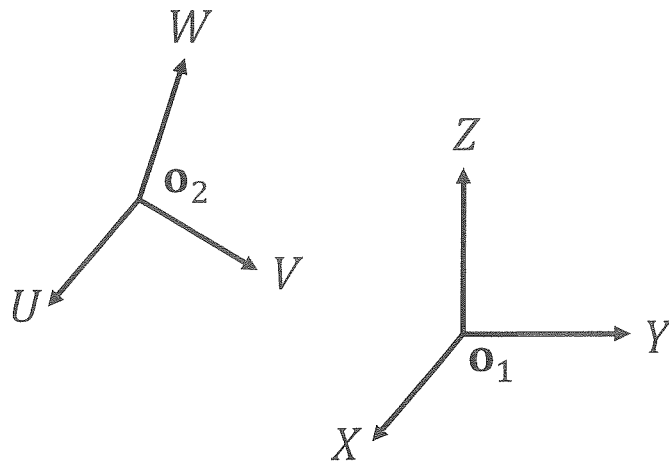
[2 points] (a) Find the rank of A .

[4 points] (b) Find a basis of null space of A .

[4 points] (c) Find a_3 and a_4 .

10. The homogeneous coordinates map $[x \ y \ z]^T$ to $[x \ y \ z \ 1]^T$, that can help us format translation in the form of linear transformations. Consider the figure below.

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There are two coordinate systems. One is with basis $B_1 = \{e_X, e_Y, e_Z\}$ and origin o_1 , and the other is with basis $B_2 = \{e_U, e_V, e_W\}$ and origin o_2 . Both B_1 and B_2 are orthonormal bases and obey the right-handed convention. The relation between B_1 and B_2 is a rotation operation.

[3 points] (a) If $[e_X]_{B_2} = [1 \ 0 \ 0]^T$ and $[e_Y]_{B_2} = [0 \ \frac{1}{2} \ \frac{\sqrt{3}}{2}]^T$, please find $[e_Z]_{B_2}$.

[4 points] (b) Without considering translation, i.e., $o_1 = o_2$, please find a 3×3 rotation matrix that can transform B_2 coordinates to B_1 coordinates based on (a).

[3 points] (c) If $[o_1 - o_2]_{B_2} = [1 \ 2 \ 1]^T$, please find $[o_1 - o_2]_{B_1}$ based on (a).

[5 points] (d) Find the linear transformation in the homogeneous coordinate system to transform $[u \ v \ w \ 1]^T$ to $[x \ y \ z \ 1]^T$ based on (a) and (c). In details, please find the 4×4 matrix in the following equation.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 1 \\ 1 & 1 \end{bmatrix}$, and $E = \begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}$.

[2 points] (a) A matrix M is symmetric if $M^T = M$. List matrices that are symmetric.

[2 points] (b) A matrix M is Hermitian if $M^H = M$. List matrices that are Hermitian.

12. [6 points] Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. Find a unitary matrix U to diagonalize A , i.e., $D = U^{-1}AU$. Please must arrange the eigenvalues in descending order in the matrix D .