

國立陽明交通大學 110 學年度碩士班考試入學試題

科目：線性代數與離散數學(1102)

考試日期：110 年 2 月 3 日 第 2 節

系所班別：資訊聯招

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【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

1. (5 points) Consider the following real number:

0.02468101214161820...

The 1st digit after the decimal point is 0; the second digit is 2; the 13th digit is 6; etc. What is the 1,000th digit after the decimal point?

2. (15 points)

- (a) (5 points)

Let $P = \{2, \{(2)\}, \{(2), (2)\}, (2), (2, 2), \{2, (2), (2), 2\}, \{(2, 2), (2, 2), ((2), 2)\}\}$.

How many elements are there in 2^P (which is the set of all subsets of P)?

- (b) (5 points) What is the definition of an infinite set W ?

- (c) (5 points) Let N be the set of natural numbers. Assume that we already know N is an infinite set. Let $S = \{k/2^n \mid n = 0, 1, 2, \dots; k \in N\}$. Note that / in this question is the division of real numbers. For example, $5/2 = 2.5$. Prove that S is an infinite set strictly according to your definition of an infinite set.

3. (5 points) Assume $n \geq 2$. Let $A = \{x_1, x_2, \dots, x_n\}$ be a set of (not necessarily distinct) natural numbers. Let $g = \gcd(x_1, x_2, \dots, x_n)$ and $l = \text{lcm}(x_1, x_2, \dots, x_n)$. Assume n , g , and l are appropriate fixed integers. Let $p = x_1 \cdot x_2 \cdot \dots \cdot x_n$, the product of x_i 's. What is the largest possible value of p in terms of n , g , and l ? What is the smallest possible value of p in terms of n , g , and l ?

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4. (5 points) How many integers between 1000 and 5000 are not divisible by 4, 5, or 6?
5. (5 points) Let a_k be the number of integer solutions of $x_1 + x_2 + x_3 = k$ where $x_1 \geq 2$, $1 \leq x_2 \leq 5$, and $0 \leq x_3 \leq 3$. Show the generating function for $\{a_k\}$.
6. (5 points) Let R_1 and R_2 be equivalence relations on the set S . Prove that $R_1 \cap R_2$ is an equivalence relation.
7. Please answer TRUE or FALSE for the following statements and give CONCISE explanations.
 - (a) (2 points) A simple graph is called regular if every vertex of this graph has the same degree. An n -cube, denoted by Q_n , is a graph that has vertices representing the 2^n bit strings of length n . Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. Q_n is regular.
 - (b) (2 points) Let V_n and E_n respectively be the number of vertices and edges of Q_n , the n -cube. Then, $E_1 = 1$, $E_2 = 4$, and $E_{n+1} = 2V_n + E_n$.
 - (c) (2 points) The complete bipartite graphs $K_{2,3}$ and $K_{3,3}$ are planar.
 - (d) (2 points) There are exactly 2 shortest paths between v_1 and v_2 in the graph represented by the adjacency matrix

$$\begin{array}{c}
 \\
 \\
 \\
 \\
 \\
 \\
 \end{array}
 \begin{array}{ccccccc}
 e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
 \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 0 & 0
 \end{bmatrix}
 \end{array}$$

with $|e_1| = 1$, $|e_2| = 2$, $|e_3| = 1$, $|e_4| = 4$, $|e_5| = 1$, $|e_6| = 4$, $|e_7| = 5$.

- (e) (2 points) In the graph given by the previous question, there exists an Euler path from v_5 to v_1 .

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8. (10 pt) True or false: NO need to justify your answer. No credit for unanswered questions. Incorrect answers will deduct the credit until zero point is earned in this section.

- (a) (correct: 1; incorrect: -1; unanswered: 0) For any square real matrix A , $A^T + A$ is always symmetric.
- (b) (correct: 1; incorrect: -1; unanswered: 0) For a square real matrix A , if $Ax = 0$ has a non-zero solution, then A is invertible.
- (c) (correct: 1; incorrect: -1; unanswered: 0) $\{(0, 0, 0)\}$ is the orthogonal complement of $\mathbb{R}^3$ in $\mathbb{R}^3$.
- (d) (correct: 1; incorrect: -1; unanswered: 0) Let A be a $m \times n$ real matrix, where $m < n$, with full row rank. $Ax = b$ has either 0 or 1 solution.
- (e) (correct: 1; incorrect: -1; unanswered: 0) Let v be a vector in S and u be a vector such that $\langle u, v \rangle = 0$, then $u \in S^\perp$.
- (f) (correct: 1; incorrect: -1; unanswered: 0) Let A be a symmetric matrix. $C(A^T)$ and $N(A^T)$ are orthogonal complement.
- (g) (correct: 1; incorrect: -1; unanswered: 0) Let $u_1, \dots, u_n$ be linearly dependent. Then $u_1, \dots, u_{n-1}$ are linearly dependent.
- (h) (correct: 1; incorrect: -1; unanswered: 0) Let U be a subspace of a vector space V . Then $(U^\perp)^\perp = U$.
- (i) (correct: 1; incorrect: -1; unanswered: 0) The null space of A is equal to the column space of $A^T A$.
- (j) (correct: 1; incorrect: -1; unanswered: 0) Let A and B be $n \times n$ real matrices. Then, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

9. (15 pt) Multiple choice questions and fill-in-the-blank questions. NO need to justify your answer. No credit for unanswered questions. Incorrect answers will deduct the credit until zero point is earned in this section.

- (a) (correct: 3; incorrect: -1; unanswered: 0) Let W be a subspace of $\mathbb{R}^n$ and $W^\perp$ denotes its orthogonal complement. If W_1 is a subspace of $\mathbb{R}^n$ such that $x \in W_1$, then $x^T u = 0$ for all $u \in W^\perp$. Justify whether the following statements are true or false.
 - (i) $\dim(W_1^\perp) \leq \dim(W^\perp)$
 - (ii) $\dim(W_1^\perp) \leq \dim(W)$
 - (iii) $\dim(W_1^\perp) \geq \dim(W)$
 - (iv) $\dim(W_1^\perp) \geq \dim(W^\perp)$
- (b) (correct: 3; incorrect: -1; unanswered: 0) Let A be a 7×5 matrix with $\text{rank}(A) = 5$. Justify whether the following statements are true or false.
 - (i) There exists at least one $b \in \mathbb{R}^7$ such that $Ax = b$ has infinite number of least square solutions.
 - (ii) For any $b \in \mathbb{R}^7$, $Ax = b$ has infinite number of solution.
 - (iii) There exists at least one $b \in \mathbb{R}^7$ such that $Ax = b$ has a unique least square solution.
 - (iv) For any $b \in \mathbb{R}^7$, $Ax = b$ has a unique solution.

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- (c) (correct: 3; incorrect: -1; unanswered: 0) Let $\hat{\mathbf{B}} = \begin{bmatrix} (i) \\ (ii) \end{bmatrix} \in \mathbb{R}^2$ be the least-squares solution that minimize $\|\mathbf{Y} - \mathbf{XB}\|^2$ where

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

Find the value for (i) and (ii).

- (d) (correct: 3; incorrect: -1; unanswered: 0) Let $\begin{bmatrix} (i) \\ (ii) \end{bmatrix}$ be the orthogonal projection of $\mathbf{b}$ onto $\mathbf{u}$, where

$$\mathbf{b} = \begin{bmatrix} -24 & 2 \\ -10 & -10 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 2 \\ -10 \end{bmatrix},$$

Find the value for (i) and (ii).

- (e) (correct: 3; incorrect: -1; unanswered: 0) Let $\begin{bmatrix} (i) \\ (ii) \\ (iii) \end{bmatrix}$ be the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, where

$$\mathbf{y} = \begin{bmatrix} 12 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Find the value for (i), (ii), and (iii).

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10. (25 points)

a. Determine.

(i) (1 points) True (T) or false (F):

If A is an invertible matrix, then $\det(A^{-1}) = [\det(A)]^{-1}$.

(ii) (1 points) True (T) or false (F):

If M is an $n \times n$ matrix and can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A, B, C, O are square matrices, and O is all zero matrix. Then $\det(M) = \det(A) \cdot \det(C)$.

(iii) (3 points) Find the determinant of the following matrix:

$$\begin{bmatrix} 2 & 0 & -1 & 1 & 3 \\ 0 & -1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 & 3 \\ 2 & -1 & -1 & -3 & 3 \\ 0 & 2 & 0 & 3 & 1 \end{bmatrix}$$

b. (10 points) Given a sequence 7, -6, 20, -24, 64, -96, ... comes from $G_{k+3} = -2G_{k+2} + 2G_{k+1} + 4G_k$ for $k \geq 0$, with $G_0 = 7$, $G_1 = -6$ and $G_2 = 20$. Please find the number G_{100} .

c. (10 points) Prove **generally** that eigenvectors of an $n \times n$ matrix corresponding to distinct eigenvalues are linearly independent. Notice that the number of distinct eigenvalues k is not necessarily equal to n (i.e. a general case with $k \leq n$). (Please leave it blank if you don't know the correct answer, or you will get **at most minus 5 points** for the wrong answer. 不會寫請留白, 答錯最多倒扣 5 分, 扣至本題組 0 分為止。)