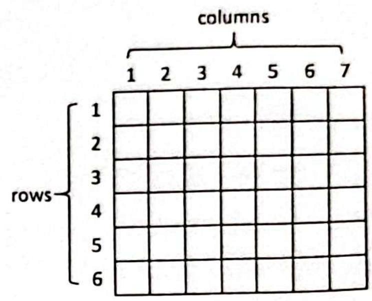
【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. **(11 points)** Consider the classic Connect Four game, where players choose a color and then take turns dropping colored tokens into a six-row, seven-column vertically suspended grid. The objective of the game is to be the first to form a horizontal, vertical, or diagonal line of four of one's own tokens. Suppose that two players, one with yellow and the other with red tokens, are playing the game. Propositional functions $Y(i,j)$, $R(i,j)$, and $B(i,j)$ assert that the square in the i-th row and the j-th column (refer to the figure) contains a yellow, a red, and no token, respectively. **Please use these predicates together with the connectives permitted to translate the statements in (a) to (c) into logic expressions.** Permitted connectives are $\vee$, $\wedge$, $\neg$, $\rightarrow$, and the pair of parentheses "()". For any propositional function $f(x,y)$, you should use the notation $\bigvee_{i=m}^{n} f(i,j)$ for $f(m,j) \vee f(m+1,j) \vee \cdots \vee f(n,j)$ and the notation $\bigwedge_{i=m}^{n} f(i,j)$ for $f(m,j) \wedge f(m+1,j) \wedge \cdots \wedge f(n,j)$.



   (a) **(4 points)** Four squares in the 2nd row contains four red tokens in a row.
   (b) **(4 points)** Each column contains at least a yellow token.
   (c) **(3 points)** If the square in the i-th row and the j-th column contains a token, the square contains either a red token or a yellow token but not both.

2. **(14 points)**
   (a) **(4 points)** Given a domain D and two predicates P and Q, let R be the truth set of $P(x)$ and S be the truth set of $Q(x)$. Pick up all the following statements that are always true? ① $|R| \leq |D|$. ② If $\forall x \in D, P(x) \rightarrow Q(x)$, then $R \subseteq S$. ③ If $R \cap S = \emptyset$, then $P(x) \vee Q(x) = \mathrm{T}$. ④ If $|R| \leq |S|$, then $\forall x \in D, P(x) \rightarrow Q(x)$. ⑤ If $R \subseteq S$, then $\forall x \in D, \neg P(x) \vee Q(x)$.
   (b) **(10 points)** Let $\aleph_0$ denote the cardinality of a countable infinite set. Name two infinite sets and use the inclusion-exclusion principle to show that $\aleph_0+\aleph_0=\aleph_0$ (Disregard whether the principle holds for infinite sets)

3. **(12 points)** Assigning 8 distinct tasks to 3 identical servers, where every task must be assigned to exactly one server.
   (a) **(4 points)** Suppose each server can handle any number of tasks, including none. How many ways can the 8 tasks be assigned to the 3 servers with no restrictions?
   (b) **(2 points)** Suppose that each server must handle at least one task. How many ways can the tasks be assigned?
   (c) **(2 points)** Suppose that each server must handle at least one task, and no server can handle 5 tasks or more, how many valid task assignments are possible?
   (d) **(4 points)** Now, consider that the servers are labeled (i.e., no longer identical), and each server must handle at least one task. Additionally, no server can handle 5 tasks or more. How many valid task assignments are there in this case?

4. **(7 points)** Consider the set $A = \{a,b,c,d,e\}$ and the relation R on A defined as follows: $R = \{(a,b),(b,c),(c,d),(d,e),(a,e)\}$.
   (a) **(1 point)** Is R reflexive?
   (b) **(1 point)** Is R symmetric?
   (c) **(1 point)** Is R transitive?
   (d) **(4 points)** Does R define a partial ordering? If not, what is the minimal number of ordered pairs that must be added to R to make it a partial order?

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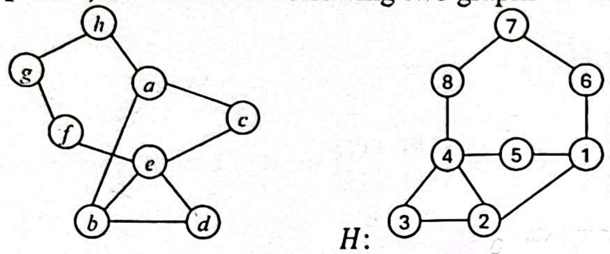
5. **(6 points)** Consider the following two graphs G and H:



G: H:

(a) **(2 points)** Are graphs G and H isomorphic?

(b) **(4 points)** If they are isomorphic, please provide the isomorphism function (a mapping from the vertices of G to the vertices of H that preserves adjacency). If they are not, explain in detail why they are not isomorphic.

6. **(25 points)** Answer the following problems. Any negative total score will be adjusted to 0 points.

Consider the problems (a)-(d), determine whether each statement is True or False. If a statement is False, provide a justification explaining why it is incorrect. You will receive 4 points for each correct answer, a penalty of -4 points for each incorrect answer, or 0 points for no answer.
For the problems (e)-(g), find the solutions and provide your answers. You will receive 3 points for each correct answer, or 0 points for each incorrect answer or no answer.

(a) A square matrix $\mathbf{A}$ is invertible if and only if $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

(b) If $\boldsymbol{W}$ is a subspace of $\mathbb{R}^n$, then the orthogonal complement of $\boldsymbol{W}$ is also a subspace of $\mathbb{R}^n$.

(c) If $\mathbf{E}$ is an $m \times m$ elementary matrix and $\mathbf{A}$ is an $m \times n$ matrix, then the null space of $\mathbf{EA}$ is the same as the null space of $\mathbf{A}$.

(d) If $f = \cos^2 x$, $g = \sin^2 x$, and $h = 1 + x^2$, then h lies in the space spanned by f and g.

(e) Find the number of parameters in the general solution of $\mathbf{Ax} = \mathbf{0}$ if $\mathbf{A}$ is a 5×7 matrix of rank 3.

(f) Find the rank of a 5×6 matrix $\mathbf{A}$ for which $\mathbf{Ax} = \mathbf{0}$ has a two-dimensional solution space.

(g) Find the value of k such that the vector $(k, 21, -12)$ is a linear combination of vectors $(-2, 2, 1)$ and $(2, 1, -2)$.

7. **(5 points)** If $*$'s denote non-zero numbers, how many non-zero product terms in the expansion of the determinant

$$\begin{vmatrix} * & 0 & * & * & 0 & * \\ 0 & 0 & * & * & 0 & * \\ 0 & * & * & * & * & 0 \\ 0 & * & * & 0 & * & 0 \\ 0 & * & * & * & * & 0 \\ * & 0 & * & * & 0 & * \end{vmatrix}.$$

8. **(10 points)** Find the orthogonal matrix $\mathbf{Q}$ in which $q_{11} > 0$ and $q_{22} > 0$, and the diagonal matrix $\mathbf{D}$ such that

$$[x \quad y]\mathbf{QDQ}^T \begin{bmatrix} x \\ y \end{bmatrix} = 6x^2 + 24xy - y^2.$$

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9. **(10 points)** Find the matrix representation of the linear transformation

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 3y \\ x + y \\ -3x + 2y \end{bmatrix}$$

with respect to bases $B_1 = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}$.