國立陽明交通大學

115 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(8102)

系所班別：資訊聯招

考試日期：115 年 2 月 4 日 第 2 節

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＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

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1. True or False: This part consists of true-or-false questions. In each case, please answer true if the statement is always true and false otherwise. And give a counterexample for any false conjecture.

1.1. **(5 points)** A *nxn* matrix **A** is said to be nonsingular,

(1) if **A** is non-invertible.

(2) if there exists a matrix **B** such that **AB**=**BA**=**I**.

(3) if the determinates of **A** in nonzero.

(4) if **Ax**=**0** where x is nonzero vector.

1.2. **(5 points)** There are many types of elementary matrices. Which of the following definitions are elementary matrices?

(1) A *nxn* matrix is obtained by interchanging two rows of **I**.

(2) A *nxn* matrix is obtained by multiplying a row of **I** by a nonzero constant.

(3) A *nxn* matrix is obtained from **I** by adding a multiple of on row to another row.

(4) A *nxn* matrix is form from the **I** by reordering its column.

1.3. **(5 points)** If there are two *nxn* matrices **A**, **B** and one scalar *r*,

(1) det(**A**) = det(**B**) implies **A** = **B**.

(2) det(r**A**+**B**) = r det(r**A**)+det(**B**).

(3) det(($\mathbf{AB}$)^T) = det(**A**)det(**B**).

(4) det(**A**) = det(**B**) where **A** and **B** are row equivalent matrices.

1.4. **(5 points)** The $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n column vectors in R^n and let $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$.

(1) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be linearly independent if and only if **A** is singular.

(2) The $\mathbf{Ax} = \mathbf{0}$ if and only if **x** is trivial $[0\ 0 \ldots 0]^T$.

(3) The rank of **A** plus the nullity of A equals n.

(4) The rank of **A** equals the rank of $\mathbf{A}^T$.

2. **(5 points)** By trial and error, find the general condition of 2×2 matrices **A**, such that $\mathbf{A}^2 = -\mathbf{I}$ with real entries.

3. **(5 points)** Find the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 4 \\ 30 & 1 & 5 & 0 & 0 & 9 \\ 3 & 5 & 3 & 8 & 0 & 4 \\ 2 & 0 & 8 & 0 & 0 & 7 \\ 7 & 0 & 4 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$.

4. **(8 points)** Find a Singular Value Decomposition of the matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$ and verify your answer.

5. **(12 points)** Let A be an mxn matrix and x be an nx1 vector. (a) (4 points) Given $\|x\| = 1$, what is the least square solution for $Ax = 0$? (b) (8 points) Please prove your answer. (Please leave it blank if you don't know the correct answer, or you will get **at most minus 5 points** (until questions 3, 4, and 5 are 0 points) for the wrong answer. 不會寫請留白，答錯最多倒扣 5 分，扣至 3, 4, 5 題 0 分為止。)

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6. There are five independent subproblems in this question.

(6.1) (5 points) Is the following statement a tautology, a contradiction, or neither? You need to provide a rigirous explanation or proof.

$\nexists a.\forall b(b \in a \leftrightarrow b \notin b)$

(6.2) (5 points) What is x in the following equation? (Assume the left-hand-side term of the equation converges.)

$$\Sigma_{r=2}^{\infty}(2^{x-r}) = \sqrt{1 + 3 \times 2^{x-2}}$$

You need to show how to obtain your answer.

(6.3) (5 points) What is the integer part of $\sqrt{5}+\sqrt{6}+\sqrt{7}+\sqrt{8}+\sqrt{9}+\sqrt{10}+\sqrt{11}+\sqrt{12}+\sqrt{13}$? You need to show how to obtain your answer.

(6.4) (5 points) Let $T =_{def} 11\ldots11$ (there are 243 1's). Is T divisible by 243? You need to explain your answer.

(6.5) (5 points) Given a strictly increasing function $f : N \to N$ so that $f(f(x)) = 2x + 1$, what is $f(6)$? Note that N denotes the set of natural numbers.

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7. For $n \geq 0$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences defined by the initial conditions $x_0 = 1$, $y_0 = 0$, and $z_0 = 0$. For $n \geq 1$, they satisfy $x_n = x_{n-1} + y_{n-1} + z_{n-1}$, $y_n = x_{n-1} + z_{n-1}$, and $z_n = y_{n-1}$. You must show all necessary steps. **Answers without justification will receive no points.**

(a) (**3 points**) Let $u_n = x_n + y_n + z_n$. Use mathematical induction to prove that $u_n = 2^n$ for all $n \geq 0$.

(b) (**3 points**) Using the result from part (a), derive a recurrence relation for y_n in terms of y_{n-1} and show that $y_n = 2^{n-1} - y_{n-1}$ for all $n \geq 1$.

(c) (**4 points**) Solve the recurrence in part (b) to obtain a closed-form expression for y_n, and verify that your formula satisfies the initial condition.

8. Define relations R and S on $\mathbb{N} = \{0,1,2,\ldots\}$ by (1) xRy if and only if $x \equiv y \bmod 7$, and (2) xSy if and only if $x \equiv y \bmod 13$. For each of the following relations on $\mathbb{N}$, determine whether it is an equivalence relation. If it is, give a proof. If it is not, explain which property of an equivalence relation fails. In each case, justify your answer. **Answers without justification will receive no points.**

(a) (**4 points**) $R \cap S$

(b) (**4 points**) $R \cup S$

9. (**7 points**) A fully connected communication network on n nodes is modeled by the complete graph K_n, where $n \geq 5$. Let $S \subseteq V(K_n) \cup E(K_n)$ be a set of failed nodes and links with $|S| \leq n-3$, such that no failed link is incident to a failed node. Let $K_n - S$ denote the network obtained after removing all failed nodes (and their incident links) and all failed links. Prove by induction on n that the remaining network $K_n - S$ contains a Hamiltonian cycle. **You may not assume any characterization theorem for Hamiltonian graphs.**