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科目：數學
節次： 4

國立臺灣大學104學年度碩士班招生考試試題

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請在答案卷上可作答部份中的第一頁繪製表格如下，並將各題答案填入表格。

1		2		3	
4		5		6	
7		8		9	
10		11			

1. (5%) Which of the following are equivalent to $p \to q$? _____

   (a) $\neg p \vee q$

   (b) $\neg p \to \neg q$

   (c) $q \to p$

   (d) $\neg q \to \neg p$

   (e) $\neg q \to p$

2. (5%) The maximum flow from node 12 to node 1 of the following network is _____.
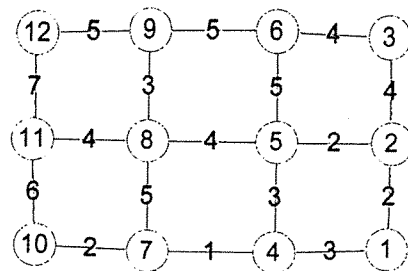
3. (10%) Simplify $\sum_{i=0}^{n} \binom{n}{i} 2^i$: _____.

4. (10%) The Euler function $\phi(120) =$ _____.

5. (10%) The number of nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_n = r,$$

   where $1 \le x_1 < x_2 < \cdots < x_n \le r$ is _____.

6. (10%) Solve the recurrence equation $a_n = 2 \times a_{n-1} + 3^{n-1}$ with $a_0 = 2$ for a_n.

見背面

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7. (10%) Alice writes down the following function f in her email to Bob and claims that f is an inner-product function for the vector space consisting of all 2×2 complex matrices:

$$f(X,Y) = \text{tr}(XY).$$

Specifically, for any 2×2 matrices X and Y whose elements are complex numbers, the value of $f(X,Y)$ is the trace of the product of X and Y. Bob immediately notices that there must be something wrong with Alice's email. What can be the possible reason or reasons for Bob's observation?

(a) Bob notices that f has this nice property that $f(X,Y) = f(Y,X)$ holds for all X and Y. This nice property prevents f from being a legal inner-product function for a vector space over complex numbers.

(b) Bob notices that there are X and Y such that $f(X,Y)$ is a complex number. Since inner-product functions are defined to measure distances, lengths, and areas, the values of $f(X,Y)$ are not allowed to be complex numbers.

(c) In order for f to be a legal inner-product function, for any matrix Y, the function $g_Y(X) = f(X,Y)$ has to be a linear function. Bob notices that there is a matrix Y such that g_Y is not linear.

(d) In order for f to be a legal inner-product function, for any matrix X, the function $h_X(Y) = f(X,Y)$ has to be a linear function. Bob notices that there is a matrix X such that h_X is not linear.

8. (10%) Let

$$A = \begin{pmatrix} -1 & 1 & 3 & -1 & 0 \\ 3 & -1 & -5 & 1 & -6 \\ 1 & 0 & -1 & 2 & 1 \\ -2 & 1 & 4 & -1 & 3 \end{pmatrix}.$$

What is the maximum rank of all 5×5 real matrices B such that AB is the 4×5 zero matrix?

9. (10%) What is the sum of the eigenvalues of the following matrix?

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}$$

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10. (10%) Consider the vector space V over real numbers that is spanned by three 2×2 matrices A_1, A_2, and A_3, where

$$A_1 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \text{ and } A_3 = \begin{pmatrix} -1 & -9 \\ 5 & -1 \end{pmatrix}.$$

Let

$$B_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix} \text{ and } B_2 = \frac{1}{3\sqrt{2}} \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}.$$

Find a matrix B_3 such that B_1, B_2, and B_3 form an orthonormal basis of V with respect to the standard inner product for V defined by

$$\left\langle \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \right\rangle = c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4.$$

11. (10%) Let V be the vector space consisting of all 2×2 real matrices. Let β be the standard ordered basis consisting of $\beta_1, \ldots, \beta_4$ in that order, where

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The function f defined by

$$f(X, Y) = \text{tr}(X) \cdot \text{tr}(Y)$$

for any matrices X and Y in V is known to be in bilinear form. What is the matrix representation of the function f in bilinear form with respect to the ordered basis β? (The answer is a 4×4 real matrix. That is why you need a larger cell for this problem in the table for your answers on the first page of your answer sheet.)

試題隨卷繳回