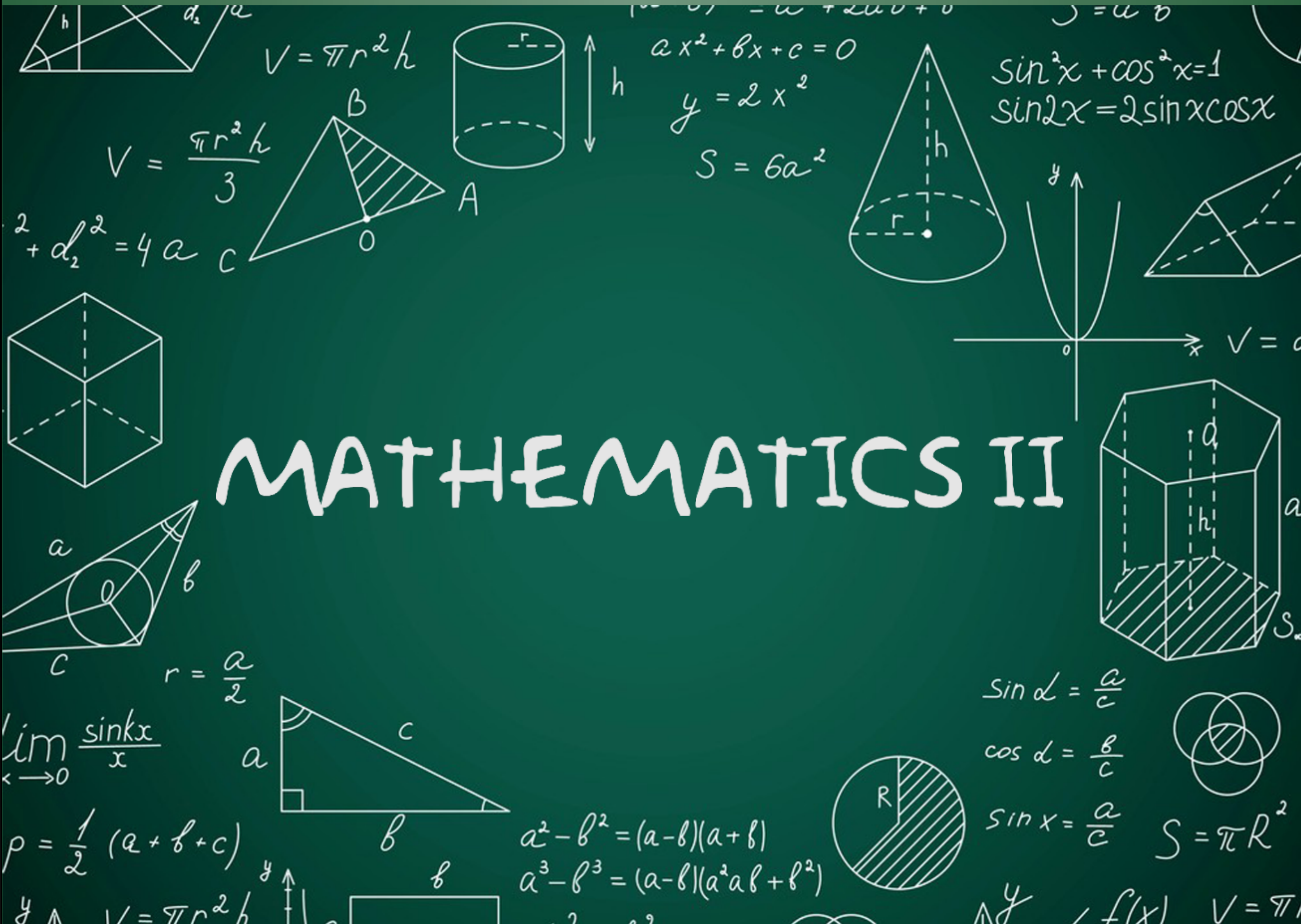
LECTURE NOTES ON

MATHEMATICS II



SATISH SHUKLA

MATHEMATICS II

Satish Shukla

To My Respected Parents

and

Teachers

Syllabus

Unit-I

Calculus of finite differences: Operators, forward difference operator, backward difference operator, E-operator, relation between them, difference of a polynomial, factorial polynomial, Inverse operator, forward difference table, Backward difference table.

Unit-II

Interpolation: Introduction to Interpolation; Interpolation with equally spaced interval, forward and backward interpolation formula, Interpolation with unequally spaced intervals, Newton divided difference interpolation, Lagrange's formula for interpolation and inverse interpolation.

Unit-III

Integral calculus: fundamental theorem of integral calculus, length of curves, volume, and surface area of revolution of curves.

Unit-IV

Evaluation of integrals using gamma function. Multiple integral: Double integral, area by double integral. Evaluation of triple integrals.

Unit-V

Linear differential equations of n^{th} order: Linear differential equations of n^{th} order, method of variation of parameter and Cauchy's homogenous linear equations.

From zero to infinite

When you look inside the "zero"... you feel like the "infinite"...

Table of Contents

UNIT-I

Calculus of finite differences: Operators, forward difference operator, backward difference operator, E-operator, relation between them, difference of a polynomial, factorial polynomial, Inverse operator, forward difference table, Backward difference table.

Calculus of finite differences

Finite Differences. In theoretical science most of the functions and relations are in explicit and continuous form. Practical problems leads us to situations when the value of function $y = f(x)$ is known not in form of an explicit formula, but value of y are known only at some points. In such cases we cannot calculate the value of y at any arbitrary given point. Similarly, in such cases it is not possible to find derivatives or integral of the function, and so, it is difficult to analyze the behaviour of function in its domain. To overcome this problem, we need the techniques of finite differences and approximation.

Let $y = f(x)$ be any function of the independent variable x. Suppose the explicit values of y in form of x is not known, but only a finite number of values of y at points $x_0, x_1, x_2 \dots, x_n$ are known and given by the following table:

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

where $y_i = f(x_i)$, $i = 0, 1, 2, \dots, n$. Then, the values of x, i.e., $x_0, x_1, x_2, \dots, x_n$ are called the argument of function and the corresponding values $y_0, y_1, y_2, \dots, y_n$ of y are called the entries. We assume that the arguments are equally spaced with space h, i.e., $x_1 = x_0 + h$, $x_2 = x_1 + h, \dots, x_n = x_{n-1} + h$. In general, $x_i = x_{i-1} + h = x_0 + ih$, $i = 1, 2, \dots, n$.

The forward difference operator Δ. It is denoted by Δ and defined by:

$$\Delta f(x) = f(x + h) - f(x).$$

By definition of Δ, it is clear that the forward difference operator finds the difference of the values of function $y = f(x)$ on two consecutive values $x + h$ and x of argument. Also:

$$\Delta y_0 = \Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0.$$

Similarly, $\Delta y_1 = y_2 - y_1, \dots, \Delta y_{n-1} = y_n - y_{n-1}$. The higher order differences are defined as follows:

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta[f(x + h) - f(x)] = \Delta f(x + h) - \Delta f(x) \\ &= f(x + 2h) - f(x + h) - [f(x + h) - f(x)] \\ &= f(x + 2h) - 2f(x + h) + f(x). \end{aligned}$$

Similarly, $\Delta^3 f(x)$ and other higher order differences can be obtain.

The differences of $y = f(x)$ for tabular values of y can be obtained by the following forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		⇨ $\Delta y_0 = y_1 - y_0$			
			⇨ $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
x_1	y_1				
		⇨ $\Delta y_1 = y_2 - y_1$		⇨ $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		⇨ $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		⇨ $\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		⇨ $\Delta y_2 = y_3 - y_2$		⇨ $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		⇨ $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		⇨ $\Delta y_3 = y_4 - y_3$			
x_4	y_4				

The backward difference operator ∇. The backward difference operator is denoted by ∇ and defined by:

$$\nabla f(x) = f(x) - f(x - h).$$

It is clear that

$$\nabla y_1 = \nabla f(x_1) - f(x_1 - h) = f(x_1) - f(x_0) = y_1 - y_0.$$

Similarly, $\nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$. Also, it is obvious that:

$$\nabla y_n = \Delta y_{n-1}.$$

The higher order backward differences can be obtained similarly. The various higher order differences can be obtained by the following backward difference table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		⇨ $\nabla y_1 = y_1 - y_0$			
			⇨ $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
x_1	y_1				
		⇨ $\nabla y_2 = y_2 - y_1$		⇨ $\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		⇨ $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		⇨ $\nabla^4 y_4 = \Delta^3 y_4 - \nabla^3 y_3$
		⇨ $\nabla y_3 = y_3 - y_2$		⇨ $\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		⇨ $\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		⇨ $\nabla y_4 = y_4 - y_3$			
x_4	y_4				

The shifting operator E. It is denoted by E and defined by:

$$Ef(x) = f(x + h).$$

The higher order shifting is defined by:

$$E^2 f(x) = Ef(x + h) = f(x + 2h).$$

Similarly, we define:

$$E^n f(x) = f(x + nh).$$

The negative powers of E is defined in similar way:

$$E^{-1}f(x) = f(x-h) \quad \text{and} \quad E^{-n}f(x) = f(x-nh).$$

Example 1. Calculate the values of backward differences of $f(4)$ from the data below:

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Solution: The difference table for the given data is as follows:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1.0				
		⇨ $\nabla f(1) = 0.5$			
1	1.5		⇨ $\nabla^2 f(2) = 0.2$		
		⇨ $\nabla f(2) = 0.7$		⇨ $\nabla^3 f(3) = 0$	
2	2.2		⇨ $\nabla^2 f(3) = 0.2$		⇨ $\nabla^4 f(4) = 0.4$
		⇨ $\nabla f(3) = 0.9$		⇨ $\nabla^3 f(4) = 0.4$	
3	3.1		⇨ $\nabla^2 f(4) = 0.6$		
		⇨ $\nabla f(4) = 1.5$			
4	4.6				

Hence, from the above table we have $\nabla f(4) = 1.5, \nabla^2 f(4) = 0.6, \nabla^3 f(4) = 0.4$ and $\nabla^4 f(4) = 0.4$. □

Relations between Δ, ∇, E and D.

Example 2. Prove the following relations:

(a) $\Delta \equiv E-1$ (b) $\nabla \equiv 1-E^{-1}$ (c) $\Delta \equiv \nabla E$ (d) $(1+\Delta)(1-\nabla) \equiv 1$

(e) $\Delta\nabla \equiv \nabla\Delta$ (f) $D \equiv \frac{1}{h}\ln(1+\Delta)$ (g) $D \equiv -\frac{1}{h}\ln(1-\nabla)$.

Solution: (a) By definition we have:

$$\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E-1)f(x).$$

Therefore:

$$\boxed{\Delta \equiv E-1} \quad \text{or} \quad \boxed{E \equiv 1+\Delta}.$$

(b)By definition we have:

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = (1-E^{-1})f(x).$$

Therefore:

$$\boxed{\nabla \equiv 1 - E^{-1}} \quad \text{or} \quad \boxed{E^{-1} \equiv 1 - \nabla}.$$

(c) By (a) and (b) we have:

$$\Delta \equiv E - 1 \equiv E - EE^{-1} \equiv (1 - E^{-1})E \equiv \nabla E.$$

(d) By (a) and (b) we have:

$$(1 + \Delta)(1 - \nabla) \equiv EE^{-1} \equiv 1.$$

(e) By (a) and (b) we have:

$$\Delta\nabla \equiv (E - 1)(1 - E^{-1}) \equiv E - EE^{-1} - 1 + E^{-1} \equiv E - 2 + E^{-1}.$$

Similarly, we have $\nabla\Delta \equiv E - 2 + E^{-1}$. Therefore, $\Delta\nabla \equiv \nabla\Delta$.

(f) By Taylor's series we know that:

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \\
\implies Ef(x) &= f(x) + hD\,[f(x)] + \frac{h^2}{2!}D^2\,[f(x)] + \frac{h^3}{3!}D^3\,[f(x)] + \cdots \\
\implies Ef(x) &= \left[1 + hD + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots\right] f(x) \\
\implies Ef(x) &= e^{hD} f(x).
\end{aligned}$$

Therefore, $E \equiv e^{hD}$, i.e., $1 + \Delta \equiv e^{hD}$ or

$$\boxed{D \equiv \frac{1}{h}\ln(1 + \Delta)}.$$

(g) Again, since $E \equiv e^{hD}$ and $E^{-1} \equiv 1 - \nabla$ we have

$$\begin{aligned}
& \frac{1}{1-\nabla} \equiv e^{hD} \\
\implies\quad & 1 - \nabla \equiv e^{-hD} \\
\implies\quad & \ln(1 - \nabla) \equiv -hD \\
\implies\quad & D \equiv -\frac{1}{h}\ln(1 - \nabla).
\end{aligned}$$

This proves the result. □

Example 3. Prove that: $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$.

Solution: We have:

$$\begin{aligned}
\text{R.H.S.} &= \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x} \\
&= \left(\frac{(E-1)^2}{E}\right) e^x \cdot \frac{Ee^x}{(E-1)^2 e^x} \\
&= \left(\frac{E^2-2E+1}{E}\right) e^x \cdot \frac{Ee^x}{(E^2-2E+1)e^x} \\
&= \left(E-2+E^{-1}\right) e^x \cdot \frac{Ee^x}{(E^2-2E+1)e^x} \\
&= \left(e^{x+h}-2e^x+e^{x-h}\right) \cdot \frac{e^{x+h}}{(e^{x+2h}-2e^{x+h}+e^x)} \\
&= \frac{e^x\left(e^{x+2h}-2e^{x+h}+e^x\right)}{(e^{x+2h}-2e^{x+h}+e^x)} \\
&= e^x \\
&= \text{L.H.S.}
\end{aligned}$$

This proves the result. □

Example 4. Prove that: $\Delta \ln f(x) = \ln\left(1+\frac{\Delta f(x)}{f(x)}\right)$.

Solution: We have

$$\begin{aligned}
\text{L.H.S.} &= \Delta \ln f(x) \\
&= (E-1)\ln f(x) \\
&= \ln f(x+h) - \ln f(x) \\
&= \ln\left(\frac{f(x+h)}{f(x)}\right) \\
&= \ln\left(\frac{Ef(x)}{f(x)}\right) \\
&= \ln\left(\frac{(1+\Delta)f(x)}{f(x)}\right) \\
&= \ln\left(\frac{f(x)+\Delta f(x)}{f(x)}\right) \\
&= \ln\left(1+\frac{\Delta f(x)}{f(x)}\right) \\
&= \text{R.H.S.}
\end{aligned}$$

This proves the result. □

Example 5. Evaluate: $\Delta\left[e^{ax}\ln(bx)\right]$.

Solution: We know that:

$$\begin{aligned}\Delta\left[e^{ax}\ln(bx)\right] &= (E-1)\left[e^{ax}\ln(bx)\right] \\ &= E[e^{ax}\ln(bx)] - e^{ax}\ln(bx) \\ &= e^{a(x+h)}\ln[b(x+h)] - e^{ax}\ln(bx)\end{aligned}$$

This is the required value. □

Example 6. Find the value of: (i) $\left(\frac{\Delta^2}{E}\right)x^3$ (ii) $\Delta^n\left(\frac{1}{x}\right)$.

Solution:

$$\text{(i)}\quad \begin{aligned}\left(\frac{\Delta^2}{E}\right)x^3 &= \left[\frac{(E-1)^2}{E}\right]x^3 \\ &= \left(E-2+E^{-1}\right)x^3 \\ &= (x+h)^3 - 2x^3 + (x-h)^3 \\ &= (x^3+3x^2h+3xh^2+h^3) - 2x^3 + (x^3-3x^2h+3xh^2-h^3) \\ &= 6xh^2.\end{aligned}$$

(ii)

$$\Delta\left(\frac{1}{x}\right) = \frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}.$$

Similarly:

$$\begin{aligned}\Delta^2\left(\frac{1}{x}\right) &= \Delta\left[\Delta\left(\frac{1}{x}\right)\right] = \Delta\left[\frac{-h}{x(x+h)}\right] \\ &= -h\left[\frac{1}{(x+h)(x+2h)} - \frac{1}{x(x+h)}\right] \\ &= \frac{(-1)^2\,2!\,h^2}{x(x+h)(x+2h)}.\end{aligned}$$

In general, we have:

$$\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)^n\,n!\,h^n}{x(x+h)(x+2h)\cdots(x+nh)}.$$

This is the required value. □

Example 7. Find the value of: (i) $\Delta \tan^{-1} x$ (ii) $\Delta^2 \cos 2x$.

Solution: (i) By the definition of forward difference operator, we know that:

$$\begin{aligned}
\Delta \tan^{-1} x &= \tan^{-1}(x+h) - \tan^{-1} x \\
&= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} \\
&= \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}.
\end{aligned}$$

(ii) By the definition of forward difference operator, we know that:

$$\begin{aligned}
\Delta^2 \cos 2x &= \Delta \{\Delta \cos 2x\} \\
&= \Delta \{\cos 2(x+h) - \cos 2x\} \\
&= \Delta \cos 2(x+h) - \Delta \cos 2x \\
&= \cos 2(x+2h) - \cos 2(x+h) - \{\cos 2(x+h) - \cos 2x\} \\
&= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h \\
&= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)] \\
&= -2\sin h\,[2\cos(2x+2h)\sin h] \\
&= -4\sin^2 h\cos(2x+2h).
\end{aligned}$$

This is the required value. □

Example 8. If $f(x) = e^{ax+b}$, then show that the leading difference from a geometric progression.

Solution: Given that $f(x) = e^{ax+b}$. Hence, by the definition of forward difference we have:

$$\begin{aligned}
\Delta f(x) &= \Delta\left[e^{ax+b}\right] = e^b \Delta\left[e^{ax}\right] \\
&= e^b\left[e^{a(x+h)} - e^{ax}\right] \\
&= \left(e^{ah}-1\right)e^{ax+b}.
\end{aligned}$$

Again, the second difference:

$$\begin{aligned}
\Delta^2 f(x) &= \Delta\left[\Delta e^{ax+b}\right] = \Delta\left[\left(e^{ah}-1\right)e^{ax+b}\right] \\
&= \left(e^{ah}-1\right)\Delta\left[e^{ax+b}\right] \\
&= \left(e^{ah}-1\right)\left[e^b\left\{e^{a(x+h)} - e^{ax}\right\}\right] \\
&= \left(e^{ah}-1\right)\left[e^{ax+b}\left\{e^{ah}-1\right\}\right] \\
&= \left(e^{ah}-1\right)^2 e^{ax+b}.
\end{aligned}$$

Similarly, we can obtain that $\Delta^n f(x) = \left(e^{ah} - 1\right)^n e^{ax+b}$ for all natural numbers n. Hence for any natural number r we have:

$$\Delta^r f(x) = \left(e^{ah} - 1\right)^r e^{ax+b} = \left(e^{ah} - 1\right)\left(e^{ah} - 1\right)^{r-1} e^{ax+b} = \left(e^{ah} - 1\right)\Delta^{r-1} f(x).$$

This show that the every two successive terms are in a common ratio $\left(e^{ah} - 1\right)$, and so, the differences are in a geometric progression. □

Example 9. Find the value of: (i) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$ (ii) $\Delta^n(e^x)$ with interval of difference $h = 1$.

Solution: (i) By the definition of forward difference operator, we know that:

$$\begin{aligned}
\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) &= \Delta^2\left(\frac{5x+12}{(x+2)(x+3)}\right)\\
&= \Delta^2\left(\frac{2}{x+2} + \frac{3}{x+3}\right)\\
&= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\}\\
&= \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}\\
&= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}\\
&= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\}\\
&\quad -3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}\\
&= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}\\
&= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}.
\end{aligned}$$

(ii) By the definition of forward difference operator, we know that:

$$\Delta e^x = e^{x+1} - e^x = (e-1)e^x.$$

Again, the second difference:

$$\begin{aligned}
\Delta^2 e^x &= \Delta\{\Delta e^x\} = \Delta\{(e-1)e^x\}\\
&= (e-1)\Delta e^x = (e-1)(e-1)e^x\\
&= (e-1)^2 e^x.
\end{aligned}$$

Again:

$$\begin{aligned}\Delta^3 e^x &= \Delta\left\{\Delta^2 e^x\right\}\\ &= \Delta\left\{(e-1)^2 e^x\right\} = (e-1)^2\Delta e^x\\ &= (e-1)^3 e^x.\end{aligned}$$

Similarly, $\Delta^n e^x = (e-1)^n e^x$. □

Theorem 1. Prove that the n^{th} difference of a polynomial of degree n is constant and all higher order differences are zero.

Proof. Let

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

By definition we have

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_{n-1}(x+h) + a_n\\ &\quad - \left[a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n\right]\\ &= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots + a_{n-1}[(x+h) - x]\\ &\quad +[a_n - a_n]\\ &= a_0nhx^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_{n-1} \text{ (using binomial theorem).}\end{aligned}$$

$$\begin{aligned}\text{Similarly, } \Delta^2 f(x) &= \Delta[\Delta f(x)]\\ &= \Delta[a_0nhx^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_n]\\ &= a_0n(n-1)h^2x^{n-2} + c_1x^{n-3} + c_2x^{n-4} + \cdots + c_{n-2}.\end{aligned}$$

Thus, we obtain:

$$\Delta^n f(x) = a_0n(n-1)(n-2)\cdots 1\cdot h^n = a_0n!h^n = \text{ constant.}$$

Therefore, $\Delta^{n+1} f(x) = \Delta[a_0n!h^n] = \Delta(\text{constant}) = 0$. □

Example 10. Find the value of $\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$.

Solution: We know that:

$$\begin{aligned}&(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\\ =\ & abcdx^{10} + \text{ terms containing } x^9 \text{ and lower degree of } x.\end{aligned}$$

Hence:

$$\begin{aligned}
&\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]\\
=\ &\Delta^{10}\left[abcdx^{10}+\text{ terms containing } x^9 \text{ and lower degree of } x\right]\\
=\ &abcd\Delta^{10}x^{10}+\Delta^{10}\left[\text{ terms containing } x^9 \text{ and lower degree of } x\right]\\
=\ &abcd\Delta^{10}x^{10}+0\\
=\ &abcd\cdot 1\cdot(10)!h^n.
\end{aligned}$$

Thus, $\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]=abcd\cdot(10)!h^n$. □

Example 11. Show that: $\nabla^2 y_8 = y_8 - 2y_7 + y_6$.

Solution: We know that $\nabla \equiv 1 - E^{-1}$, hence:

$$\nabla^2 \equiv \left(1-E^{-1}\right)^2 = 1-2E^{-1}+E^{-2}.$$

Therefore:

$$\begin{aligned}
\nabla^2 y_8 &= \left(1-2E^{-1}+E^{-2}\right)y_8\\
&= y_8-2E^{-1}y_8+E^{-2}y_8\\
&= y_8-2y_7+y_6.
\end{aligned}$$

This proves the required result. □

Motivation for the calculus of finite differences

How can we evaluate $\int_a^b f(x)dx$, where f is continuous in its domain? The answer is given by the the fundamental theorem of calculus. It says that if $g(x)$ is the anti-derivative of $f(x)$, i.e., $f(x)=g'(x)$ then:

$$\int_a^b f(x)dx = g(a)-g(b).$$

Obviously, the above problem is meaningful when the function f is continuous in its domain (in general). For a function $f(x)$, where the value of function is known only at some finite number of values of x in the interval $[a,b]$, an analogue of the above problem can be stated as: how can we evaluate $\sum_{x=a}^{b} f(x)$? Such problems occurs frequently in practical and theoretical calculations.

To answer this question, we need a result similar to the fundamental theorem of calculus which works for "$\sum$" instead "$\int$".

Definition 1 (Anti-difference operator)**.** A function $g(x)$ is called anti-difference of the function $f(x)$ if $\Delta g(x) = f(x)$.

Theorem 2 (Fundamental theorem of finite difference calculus)**.** *Let $g(x)$ be an anti-difference of $f(x)$. Then, $\sum_{x=a}^{b} f(x) = g(b+h)-g(a)$, $b = a+(n-1)h$.*

Proof. By definition we have:

$$\begin{aligned}
\sum_{x=a}^{b} f(x) &= \sum_{x=a}^{b} \Delta g(x) = \sum_{x=a}^{b}[g(x+h) - g(x)] = \sum_{x=a}^{b} g(x+h) - \sum_{x=a}^{b} g(x)\\
&= g(a+h) + g(a+2h) + \cdots + g(b+h) - [g(a) + g(a+h) + \cdots + g(b)]\\
&= g(a+h) + g(a+2h) + \cdots + g(a+nh)\\
&\quad -[g(a) + g(a+h) + \cdots + g(a+\overline{n-1}h)]\\
&= g(b+h) - g(a)
\end{aligned}$$

which proves the theorem. □

Next, we collect some tools for finding anti-difference of a function.

Factorial notation or falling powers: Suppose n be any integer, then the factorial power of x is denoted by $x^{(n)}$ and it is defined by:

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-\overline{n-1}h).$$

If the length of interval is assumed $h = 1$, then

$$x^{(n)} = x(x-1)(x-2)\cdots(x-\overline{n-1}).$$

Example 12. Prove that: $\Delta x^{(n)} = nx^{(n-1)}$, where $h = 1$.

Solution: By definition, we have:

$$\begin{aligned}
\Delta x^{(n)} &= (x+1)^{(n)} - x^{(n)}\\
&= (x+1)(x)(x-1)\cdots(x-\overline{n-2}) - x(x-1)(x-2)\cdots(x-\overline{n-1})\\
&= x(x-1)\cdots(x-\overline{n-2})\big[(x+1) - (x-\overline{n-1})\big]\\
&= nx(x-1)\cdots(x-\overline{n-2})\\
&= nx^{(n-1)}.
\end{aligned}$$

This proves the result. □

Example 13. Express $y = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation and hence show that $\Delta^3 y = 12$.

Solution: Suppose

$$y = 2x^3 - 3x^2 + 3x - 10 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D.$$

To find the constants A, B, C, D, we use the synthetic division as follows:

	x^3	x^2	x	constant
1	2 +0	−3 +2	3 −1	$-10 = D$
2	2 +0	−1 +4	$2 = C$	
3	2 +0	$3 = B$		
	$2 = A$			

Therefore

$$y = 2x^3 - 3x^2 + 3x - 10 = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10.$$

Also,

$$\begin{aligned}\Delta^3 y &= \Delta^3[2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10] \\ &= \Delta^2[6x^{(2)} + 6x^{(1)} + 2] \\ &= \Delta[12x^{(1)} + 6] \\ &= 12.\end{aligned}$$

Hence, $\Delta^3 y = 12$. □

Example 14. Find the function whose first forward difference is $6x^2 + 2$.

Solution: Suppose $f(x)$ is the function whose first forward difference is $6x^2 + 2$, i.e.,

$$\Delta f(x) = 6x^2 + 2 = Ax^{(2)} + Bx^{(1)} + C.$$

To find the constants A, B, C, we use the synthetic division as follows:

	x^2	x	constant
1	6 +0	0 +6	$2 = C$
2	6 +0	$6 = B$	
3	$6 = A$		

Therefore

$$\Delta f(x) = 6x^{(2)} + 6x^{(1)} + 2.$$

Integrating the above we obtain:,

$$f(x) \quad = \quad 2x^{(3)} + 3x^{(2)} + 2x^{(1)} + c.$$

This is the required function. □

Exercise (Assignment)

(Q.1) Prove that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.
Hint: Use the relation $\Delta \equiv E - 1$ and $E^n y_i = y_{i+n}$, $n = 1, 2, 3$.

(Q.2) Prove that $\Delta\left(\frac{1}{f(x)}\right) = -\frac{\Delta f(x)}{f(x)f(x+1)}$, assume $h = 1$.
Hint: Think.

(Q.3) Evaluate $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$, assume $h = 1$.
Ans: $\frac{10x+32}{(x+2)(x+3)(x+4)(x+5)}$.

(Q.4) Evaluate $\Delta^3\left[(1-x)(1-2x)(1-3x)\right]$, assume $h = 2$.
Ans: -288.

(Q.5) Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$.
Hint: Use the relations between Δ, ∇ and E.

(Q.6) Construct the table of differences for the data below:

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

(Q.7) Express $x^3 - 2x^2 + x - 1$ into factorial polynomial hence show that $\Delta^4 f(x) = 0$.
Ans: $f(x) = x^{(3)} + x^{(2)} - 1$.

(Q.8) Represent the function $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$ and all its successive differences into factorial notation. Hence show that $\Delta^5 f(x) = 0$.
Ans: $f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$.

(Q.9) Find the function whose first forward difference is $2x^3 + 3x^2 - 5x + 4$.
Ans: $f(x) = \frac{1}{2}x^{(4)} + 3x^{(3)} + 4x^{(1)} + c$.

(Q.10) Find the function whose first forward difference is $9x^2 + 11x + 5$.

Finding the missing terms in a given series

In this section, we deal with the data in which few terms are missing and we have to recover those missing values. We know that to fit a straight line we must have two points i.e., two points known means we can assume that a first degree curve can be fitted. Generally, n points known means a $(n-1)^{\text{th}}$ degree curve can be fitted with the given data. Then we apply the theorem that says the n^{th} difference of a $(n-1)^{\text{th}}$ degree polynomial is zero.

Example 15. Find the missing value in table given below:

x:	0	1	2	3	4
y:	1	3	9	?	81

Explain why the value differ from 3^3 or 27.

Solution: In the above data, there are 4 points are known (as their both x and y co-ordinates are known). So, we can assume that y is a third degree polynomial. Hence all the fourth differences must be zero. Let a be the unknown value of y. Then the difference table will be as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		⇨ 2			
1	3		⇨ 4		
		⇨ 6		⇨ $a-19$	
2	9		⇨ $a-15$		⇨ $124-4a$
		⇨ $a-9$		⇨ $105-3a$	
3	a		⇨ $90-2a$		
		⇨ $81-a$			
4	81				

Since the fourth difference must be zero, we have $124-4a=0 \implies a=31$. This value is not $3^3=27$, because we assume y a polynomial of degree three in x, while the function is actually $y=3^x$, an exponential function. □

Example 16. Find the missing values in table given below:

x:	0	1	2	3	4	5	6	7
y:	1	-1	1	-1	1	?	?	?

Solution: In the above data, there are total 8 points are given. But, only for 5 points the value of y are known for given values of x. So, we can assume that y is a fourth degree polynomial. Hence, its fourth difference will be constant and the fifth differences must be zero. Let corresponding to $x = 5, 6, 7$ the values of y are a, b, c respectively. Then, the difference table for the given data is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		-2				
1	-1		4			
		2		-8		
2	1		-4		16	
		-2		8		$a-31$
3	-1		4		$a-15$	
		2		$a-7$		$b-5a+26$
4	1		$a-3$		$b-4a+11$	
		$a-1$		$b-3a+4$		$c-5b+10a-16$
5	a		$b-2a+1$		$c-4b+6a-5$	
		$b-a$		$c-3b+3a-1$		
6	b		$c-2b+a$			
		$c-b$				
7	c					

Since the fifth difference must be zero, we have:

$$\begin{aligned} a - 31 &= 0; \\ b - 5a + 26 &= 0; \\ c - 5b + 10a - 16 &= 0. \end{aligned}$$

The first equation of the above gives $a = 31$. Putting this value in the second equation we get:

$$\begin{aligned} & b - 155 + 26 = 0 \\ \implies & b = 129. \end{aligned}$$

Putting the values of a and b in the third equation we get

$$\begin{aligned} & c - 645 + 310 - 16 = 0 \\ \implies & b = 351. \end{aligned}$$

Hence, the required values are:

$$a = 31, \quad b = 129, \quad c = 351.$$ □

Example 17. If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.

Solution: Since the $\Delta^5 y_1$ =constant, therefore we must have $\Delta^6 y_1 = 0$, i.e.,

$$\begin{aligned}
& \Delta^6 y_1 = 0 \\
\implies & (E-1)^6 y_1 = 0 \\
\implies & (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)y_1 = 0 \\
\implies & y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0 \\
\implies & (y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0 \\
\implies & y_4 = \frac{1}{20}\left[(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3)\right] \\
\implies & y_4 = \frac{1}{20}[-7845 - 6 \times 686 + 15 \times 1088] = 571.
\end{aligned}$$

Hence, the required value is $y_4 = 571$. □

Example 18. If $y_{10} = 3, y_{11} = 6, y_{12} = 11, y_{13} = 18, y_{14} = 27$, then find y_4.

Solution: For the given values the backward difference table is as follows:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	$y_{10} = 3$				
		⇨ $\nabla y_{11} = 3$			
11	$y_{11} = 6$		⇨ $\nabla^2 y_{12} = 2$		
		⇨ $\nabla y_{12} = 5$		⇨ $\nabla^3 y_{13} = 0$	
12	$y_{12} = 11$		⇨ $\nabla^2 y_{13} = 2$		⇨ $\nabla^4 y_{14} = 0$
		⇨ $\nabla y_{13} = 7$		⇨ $\nabla^3 y_{14} = 0$	
13	$y_{13} = 18$		⇨ $\nabla^2 y_{14} = 2$		
		⇨ $\nabla y_{14} = 9$			
14	$y_{14} = 27$				

Now we know that:

$$\begin{aligned}
y_4 &= y_{14-10} = E^{-10} y_{14} = \left(E^{-1}\right)^{10} y_{14} \\
&= (1-\nabla)^{10} y_{14} \\
&= \left[1 - 10\nabla + \frac{10 \cdot (10-1)}{2!}\nabla^2 - \frac{10 \cdot (10-1) \cdot (10-2)}{3!}\nabla^3 + \cdots\right] y_{14} \\
&= y_{14} - 10\nabla y_{14} + \frac{10 \cdot 9}{2}\nabla^2 y_{14} - \frac{10 \cdot 9 \cdot 8}{6}\nabla^3 y_{14} + \cdots \\
&= 27 - 10 \cdot 9 + 45 \cdot 2 - 120 \cdot 0 + \cdots = 27.
\end{aligned}$$

Hence, $y_4 = 27$. □

Exercise (Assignment)

(Q.1) From the following table find the missing value:

x:	2	3	4	5	6
y:	45.0	49.2	54.1	?	67.4

Ans: 60.05.

(Q.2) From the following table find the missing value:

x:	1	2	3	4	5	6	7
y:	2	4	8	?	32	64	128

Ans: 16.1.

(Q.3) From the following table find the missing values:

x:	0	0.1	0.2	0.3	0.4	0.5	0.6
y:	0.135	?	0.111	0.100	?	0.082	0.074

Ans: $y(0.1) = 0.123, y(0.4) = 0.090$.

♣ ♣ ♣ ♣

UNIT-II

Interpolation: Introduction to interpolation; Interpolation with equally spaced interval, forward and backward interpolation formula, Interpolation with unequally spaced intervals, Newton divided difference interpolation, Lagrange's formula for interpolation and inverse interpolation.

Interpolation

The interpolation is a technique with the help of which we can construct the new data points within the range of given discrete data points. In other words, if a function $f(x)$ is unknown, but the values of this function at some discrete points, say $x_0, x_1, \dots, x_n$ are known, then we can find the value of $f(x)$ at a point $x \in [x_0, x_n]$. For this, we approximate the function $f(x)$ by a polynomial of degree maximum n (since the value of function is known at $n+1$ points). This process is called the *polynomial interpolation.*

According to the nature of points $x_0, x_1, \dots, x_n$ the process of interpolation is divided into the following:

(I) Interpolation for equally spaced intervals. In this case, the values $x_0, x_1, \dots, x_n$ are equally spaced, i.e., $x_i = x_{i-1}+h$ for $i = 1, 2, \dots, n$ and h is the space or length of the interval. For such case, we will use Newton's forward interpolation formula or Newton's backward interpolation formula. If the point at which the value is to interpolated lies in the upper half of the difference table then we use Newton's Forward interpolation formula. Newton's backward interpolation formula is used when the point at which the value is to interpolated lies in the lower half of the difference table.

(II) Interpolation for unequally spaced intervals. In this case, the values $x_0, x_1, \dots, x_n$ are not equally spaced. For such cases Newton's divided difference formula or Lagrange's interpolation formula is used.

Newton's forward interpolation formula. Suppose the value of function $y = f(x)$ is given at $n+1$ equally spaced points $x_0, x_1 = x_0+h, x_2 = x_1+h, \dots, x_n = x_{n-1}+h$, and we have to find the value of function at an intermediate point $x \in [x_0, x_n]$. Suppose $x = x_0 + rh$, i.e., $r = \dfrac{x - x_0}{h}$. Then we know that

$$\begin{aligned} y &= f(x) = f(x_0 + rh) \\ &= E^r f(x_0) = E^r y_0 \\ &= (1+\Delta)^r y_0 \\ &= \left[1 +^r C_1\Delta +^r C_2\Delta^2 +^r C_3\Delta^3 + \cdots +^r C_r\Delta^r\right] y_0. \end{aligned}$$

Therefore:

$$\boxed{y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots + \Delta^r y_0.}$$

Newton's backward interpolation formula. Suppose the value of function $y = f(x)$ is given at $n+1$ equally spaced points $x_0, x_1 = x_0 + h, x_2 = x_1 + h, \ldots, x_n = x_{n-1} + h$, and we have to find the value of function at an intermediate point $x \in [x_0, x_n]$. Suppose $x = x_n - rh$, i.e., $r = \dfrac{x_n - x}{h}$. Then we know that

$$\begin{aligned} y &= f(x) = f(x_n - rh) \\ &= E^{-r} f(x_n) = \left(E^{-1}\right)^r y_n \\ &= (1 - \nabla)^r y_0 \\ &= \left[1 -^r C_1 \nabla +^r C_2 \nabla^2 -^r C_3 \nabla^3 + \cdots + (-1)^r \, {}^r C_r \nabla^r\right] y_n. \end{aligned}$$

Therefore:

$$\boxed{y = y_n - r\nabla y_n + \frac{r(r-1)}{2!}\nabla^2 y_n - \frac{r(r-1)(r-2)}{3!}\nabla^3 y_n + \cdots + (-1)^r \nabla^r y_n.}$$

Example 19. The area A of a circle of diameter d is given by the following table:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Find the area of circle of diameter 82.

Solution: The forward difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

We represent d by x and A by y. Since $d = 82$ is near the initial value 80 we will use the forward interpolation formula. Then, for $x = 82$ we have $r = \dfrac{x - x_0}{h} =$

$\frac{82-80}{5} = 0 \cdot 4$. Now by Newton's forward interpolation formula we have:

$$\begin{aligned} y(82) &= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \\ &= 5026 + (0\cdot 4)(648) + \frac{0\cdot 4(0\cdot 4-1)}{2}(40) + \frac{0\cdot 4(0\cdot 4-1)(0\cdot 4-2)}{6}(-2) \\ &\quad + \frac{0\cdot 4(0\cdot 4-1)(0\cdot 4-2)(0\cdot 4-3)}{24}(4) \\ &= 5280.1056 \text{ sq. units.} \end{aligned}$$

This is the required value. □

Example 20. From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks:	30-40	40-50	50-60	60-70	70-80
No. of Students:	31	42	51	35	31

Solution: We construct the cumulative table which is as follows:

Marks less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		⇨ 42			
50	73		⇨ 9		
		⇨ 51		⇨ -25	
60	124		⇨ -16		⇨ 37
		⇨ 35		⇨ 12	
70	159		⇨ -4		
		⇨ 31			
80	190				

We have to find $y(45)$ and 45 is near the initial value 40, therefore we will use the Newton's forward interpolation formula. Then, for $x = 45$ we have

$$r = \frac{x - x_0}{h} = \frac{45-40}{10} = 0 \cdot 5.$$

Now by Newton's forward interpolation formula we have:

$$\begin{aligned} y(45) &= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \\ &= 31 + (0\cdot 5)(42) + \frac{0\cdot 5(0\cdot 5-1)}{2}(9) + \frac{0\cdot 5(0\cdot 5-1)(0\cdot 5-2)}{6}(-25) \\ &\quad + \frac{0\cdot 5(0\cdot 5-1)(0\cdot 5-2)(0\cdot 5-3)}{24}(37) \\ &= 47.87 \\ &\approx 48. \end{aligned}$$

Thus, the number of students obtained marks less than 45, i.e., $y(45) = 48$ and from the table the number of students obtained marks less than 40 is $y(40) = 31$. Therefore, the number of students obtaining the marks between 40 and 45 will be:

$$y(45) - y(40) = 48 - 31 = 17.$$

This is the required value. □

Example 21. Find a polynomial which takes the following values:

x:	0	1	2	3
y:	1	2	1	10

Hence or otherwise, evaluate $f(4)$.

Solution: The difference table for the given function is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		−2	
		−1		12
2	1		10	
		9		
3	10			

Here $h = 1$, $x_0 = 0$, and so

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x.$$

Now by forward interpolation formula we have:

$$\begin{aligned} f(x) &= y \\ &= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \\ &= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12) \\ &= 1 + x - x(x-1) + 2x(x-1)(x-2) \\ &= 2x^3 - 7x^2 + 6x + 1. \end{aligned}$$

Now, putting $x = 4$ in the above formula for $y = f(x)$ we obtain:

$$\begin{aligned} f(4) &= 2(4^3) - 7(4^2) + 6(4) + 1 \\ &= 41. \end{aligned}$$

Hence, $f(4) = 41$. □

Example 22. Evaluate $f(3.75)$ from the table given below:

x:	2.5	3	3.5	4	4.5	5
y:	24.145	22.043	22.225	18.644	17.262	16.047

Solution: Here $h = 0.5$. Since 3.75 is near to the final value $x = 5$ we will use the Newton's backward interpolation formula. Then,

$$r = \frac{x_n - x}{h} = \frac{5 - 3.75}{0.5} = 2.5.$$

The backward difference table is given as follows:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
2.5	24.145					
		⇨ −2.102				
3	22.043		⇨ 2.284			
		⇨ 0.182		⇨ −6.047		
3.5	22.225		⇨ −3.763		⇨ 12.009	
		⇨ −3.581		⇨ 5.962		⇨ −20.003
4	18.644		⇨ 2.199		⇨ −7.994	
		⇨ −1.382		⇨ −2.032		
4.5	17.262		⇨ 0.167			
		⇨ −1.215				
5	16.047					

Now by backward interpolation formula we have:

$$
\begin{aligned}
y &= f(3.75) \\
&= y_5 - r\nabla y_5 + \frac{r(r-1)}{2!}\nabla^2 y_5 - \frac{r(r-1)(r-2)}{3!}\nabla^3 y_5 \\
&\quad + \frac{r(r-1)(r-2)(r-3)}{4!}\nabla^4 y_5 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\nabla^5 y_5 \\
&= 16.047 - (2.5)(-1.215) + \frac{(2.5)(1.5)}{2}(0.167) - \frac{(2.5)(1.5)(0.5)}{6}(-2.032) \\
&\quad + \frac{(2.5)(1.5)(0.5)(-0.5)}{24}(-7.994) + \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{120}(-20.003) \\
&= 16.047 + 3.037 + 0.313 + 0.635 + 0.312 - 0.2352 \\
&= 20.1088.
\end{aligned}
$$

Hence, $f(3.75) = 20.1088$. □

Example 23. Find the values of $f(1.5)$ and $f(5.5)$ from the following table:

x:	0	1	2	3	4	5	6	7
y:	1	-1	1	2	12	30	45	50

Solution: The difference table is given below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$
0	$\boxed{1}$							
		$\boxed{-2}$						
1	-1		$\boxed{4}$					
		2		$\boxed{-5}$				
2	1		-1		$\boxed{15}$			
		1		10		$\boxed{-26}$		
3	2		9		-11		$\boxed{27}$	
		10		-1		1		$\boxed{-14}$
4	12		8		-10		$\boxed{13}$	
		18		-11		$\boxed{14}$		
5	30		-3		$\boxed{4}$			
		15		$\boxed{-7}$				
6	45		$\boxed{-10}$					
		$\boxed{5}$						
7	50							

Now use the forward interpolation formula for $f(1.5)$ and backward interpolation formula for $f(5.5)$. □

Exercise (Assignment)

(Q.1) Find the values of $f(2.1)$ and $f(2.4)$ from the following table:

x:	2.0	2.1	2.2	2.3	2.4	2.5	2.6
$y = f(x)$:	0.135	-	0.111	0.100	-	0.082	0.074

Ans. $f(2.1) = 0.123$ and $f(2.4) = 0.0904$.

(Q.2) Fit a polynomial to the given data:

x :	4	6	8	10
y :	1	3	8	16

Hence find y at $x = 5$.

(Q.3) Given that $\sin(45°) = 0.7071, \sin(50°) = 0.7660, \sin(55°) = 0.8192, \sin(60°) = 0.8660$. Then find $\sin(52°)$.

Hint: Use Newton's forward difference formula with $x = 52$. **Ans.** 0.788.

(Q.4) Find the number of mens getting wages between Rs. 10 and Rs. 15 from the following data:

Wages	0-10	10-20	20-30	30-40
Frequency	9	30	35	42

Ans. 15.

(Q.5) Find the cubic polynomial in x for the following polynomial:

x :	0	1	2	3	4	5
y :	-3	3	11	27	57	107

Ans. $f(x) = x^3 - 2x^2 + 7x - 3$.

(Q.6) The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $v = 15$:

v :	10	20	30	40
p :	1.1	2.0	4.4	7.9

Ans. $p(15) = 1.325$.

(Q.7) Find $f(42)$ from the following data:

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

Ans. $f(42) \approx 219$.

Interpolation with unequally spaced intervals

For unequally spaced intervals we will use two formulae: (i) The Lagrange's formula; (ii) Newton's Divided Difference formula.

(i) **Lagrange's formula.** Suppose, the values of function $y = f(x)$ at points $x_0, x_1, x_2, \ldots, x_n$ be $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$. Then, the Lagrange's approximated polynomial of degree n is given by:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_3)\cdots(x_n-x_{n-1})}y_n.$$

(ii) **Newtons divided difference formula.** First we define the divided difference of a function. Suppose $x_0, x_1, x_2, \ldots, x_n$ be the values of arguments x and $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$ be the corresponding values of y. Then the first divided difference of f is denoted by $\Delta f(x_0)$ or $f[x_0, x_1]$ and

$$\begin{aligned} \Delta f(x_0) &= f[x_0, x_1] \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \end{aligned}$$

Similarly, we define

$$\begin{aligned} \Delta^2 f(x_0) &= f[x_0, x_1, x_2] \\ &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1} \end{aligned}$$

and so on.

Suppose, the values of function $y = f(x)$ at points $x_0, x_1, x_2, \ldots, x_n$ be $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$. Then, the Newton's divided difference approximated polynomial of degree n is given by:

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) \\ &\quad + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\Delta^n f(x_0). \end{aligned}$$

Example 24. Find the Newton's divided difference approximated polynomial for the function given below and hence find $f(8), f(9)$ and $f(15)$.

x :	4	5	7	10	11	13
$y = f(x)$:	48	100	294	900	1210	2028

Solution: The divided difference table for the given function is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		$\frac{1-1}{11-4}=0$
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		$\frac{1-1}{13-5}=0$
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Therefore, the Newton's divided difference approximated polynomial will be:

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)\,\Delta\, f(x_0) \\ &\quad +(x-x_0)(x-x_1)\,\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0) \\ &\quad +(x-x_0)(x-x_1)(x-x_2)(x-x_3)\,\Delta^4 f(x_0) \\ &\quad +(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\,\Delta^5 f(x_0) \\ &= 48 + 52(x-4) + 15(x-4)(x-5) + (x-4)(x-5)(x-7). \end{aligned}$$

Thus,

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + (x-4)(x-5)(x-7). \tag{1}$$

Putting $x = 8$ in (1) we get

$$f(8) = 48 + 52(8-4) + 15(8-4)(8-5) + (8-4)(8-5)(8-7) = 448.$$

Similarly, $f(9) = 648$ and $f(15) = 3150$. □

Example 25. Given that $f(0) = -18, f(1) = 0, f(3) = 0, f(5) = -248, f(6) = 0, f(9) = 13104$, then find $f(x)$.

Solution: Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 5, x_4 = 6, x_5 = 9$. Therefore, the points are unequally spaced. We shall use the Newton's divided difference interpolation formula for the calculation of $f(x)$. The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	-18					
		$\frac{0+18}{1-0}=18$				
1	0		$\frac{0-18}{3-0}=-6$			
		$\frac{0-0}{3-1}=0$		$\frac{-31+6}{5-0}=-5$		
3	0		$\frac{-124-0}{5-1}=-31$		$\frac{31+5}{6-0}=6$	
		$\frac{-248-0}{5-3}=-124$		$\frac{124+31}{6-1}=31$		$\frac{15-6}{9-0}=1$
5	-248		$\frac{248+124}{6-3}=124$		$\frac{151-31}{9-1}=15$	
		$\frac{0+248}{6-5}=248$		$\frac{1030-124}{9-3}=151$		
6	0		$\frac{4368-248}{9-5}=1030$			
		$\frac{13104-0}{9-6}=4368$				
9	13104					

Therefore, the Newton's divided difference approximated polynomial will be:

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0) \Delta f(x_0) \\ &\quad +(x-x_0)(x-x_1) \Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2) \Delta^3 f(x_0) \\ &\quad +(x-x_0)(x-x_1)(x-x_2)(x-x_3) \Delta^4 f(x_0) \\ &\quad +(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) \Delta^5 f(x_0) \\ &= -18 + 18x - 6x(x-1) + 7.4x(x-1)(x-3) \\ &\quad +1.87x(x-1)(x-3)(x-5) + 1.63x(x-1)(x-3)(x-5)(x-6) \\ &= x^5 - 9x^4 + 18x^3 + 9x - 18. \end{aligned}$$

This is the required polynomial. □

Inverse Interpolation. Sometimes it will be required to find out the value of x corresponding to a value of y. Keeping in mind x and y are variables representing independent and dependent variable, in such case we have to treat y as independent variable and x as dependent variable so that the interpolation formulae remain valid in this case also. Since y is considered as the independent variable, we have to check whether the values of y are equally spaced or not and accordingly we have to decide which interpolation formula is applicable.

Example 26. Find the value of x for $y = 2.2$ from the following table:

x :	0	1	2	3	4	5
y :	1	2	3	5	12	30

Solution: Since the values of y are not equidistant, we use the Newton's inverse divided difference formula. Then, the divided difference table for y will be:

y	x	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1	0					
		$\frac{1-0}{2-1}=1$				
2	1		$\frac{1-1}{3-1}=0$			
		$\frac{2-1}{3-2}=1$		$\frac{-0.167-0}{5-1}=-0.042$		
3	2		$\frac{0.5-1}{5-2}=-0.167$		$\frac{0.013-0.042}{12-1}=0.005$	
		$\frac{3-2}{5-3}=0.5$		$\frac{-0.04+0.167}{12-2}=0.013$		≈ 0
5	3		$\frac{0.143-0.5}{12-3}=-0.04$		$\frac{0.001-0.013}{30-2}\approx 0$	
		$\frac{4-3}{12-5}=0.143$		$\frac{-0.003+0.04}{30-3}=0.001$		
12	4		$\frac{0.056-0.143}{30-5}=-0.003$			
		$\frac{5-4}{30-12}=0.056$				
30	5					

Therefore, by Newton's divided difference formula we have

$$\begin{aligned}
x &= x_0+(y-y_0)\,\Delta\,x_0\\
&\quad +(y-y_0)(y-y_1)\,\Delta^2\,x_0+(y-y_0)(y-y_1)(y-y_2)\,\Delta^3\,x_0\\
&\quad +(y-y_0)(y-y_1)(y-y_2)(y-y_3)\,\Delta^4\,x_0\\
&\quad +(y-y_0)(y-y_1)(y-y_2)(y-y_3)(y-y_4)\,\Delta^5\,x_0\\
&= 0+(2.2-1)(1)+(2.2-1)(2.2-2)(0)+(2.2-1)(2.2-2)(2.2-3)(-0.042)\\
&\quad +(2.2-1)(2.2-2)(2.2-3)(2.2-5)(0.005)\\
&= 1.2+0.008+0.003\\
&= 1.211.
\end{aligned}$$

This is the required value of x. □

Example 27. From the given table find for what value of x when $y=13.6$:

x :	30	35	40	45	50
y :	15.9	14.9	14.1	13.3	12.5

Solution: We will find the value $x(13.6)$ by Lagrange's inverse interpolation formula. Here $x_0=30, x_1=35, x_2=40, x_3=45, x_4=50$ and $y_0=15.9, y_1=14.9, y_2=14.1, y_3=13.3, y_4=12.5$ and $y=13.6$. Then, we have:

$$\begin{aligned}
x &= \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1\\
&\quad +\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3\\
&\quad +\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)}x_4.
\end{aligned}$$

Putting all the values we get:

$$
\begin{aligned}
x &= \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}30 \\
&\quad + \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)}35 \\
&\quad + \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-13.6)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-13.6)}40 \\
&\quad + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.6)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-13.6)}45 \\
&\quad + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.6)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.6)}50 \\
&= 43.195.
\end{aligned}
$$

This is the required value of x. □

Exercise (Assignment)

(Q.1) Use Newton's divided difference formula to find the form of $f(x)$, hence find $f(4)$:

x :	0	2	3	6
$f(x)$:	648	704	729	792

Ans. $f(x) = -x^2 + 30x + 648$.

(Q.2) Given $\log(654) = 2.8156, \log(658) = 2.8182, \log(659) = 2.8189$ and $\log(661) = 2.8202$. Find $\log(656)$.

Ans. Use Lagrange's interpolation formula $\log(656) = 2.8169$.

(Q.3) Use Lagrange's formula to find the value of $f(9)$, where:

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Ans. $f(9) = 810$.

(Q.4) Apply Lagrange's formula and find the value of x when $f(x) = 15$

x :	5	6	9	11
$y = f(x)$:	12	13	14	16

Ans. Use Lagrange's inverse interpolation formula $x(15) = 9.125$.

♣ ♣ ♣ ♣

UNIT-III

Integral calculus: fundamental theorem of integral calculus, length of curves, volume, and surface area of revolution of curves.

Riemann Integral

The idea. Suppose, f be a continuous function defined on $[a, b]$ and we want to calculate the area bounded by this function with the x-axis from point $x = a$ to point $x = b$. This area is shown by the shaded (blue) part in the Figure 1. Riemann suggested that this area can be calculated by dividing this area into small rectangles of infinitely small width.

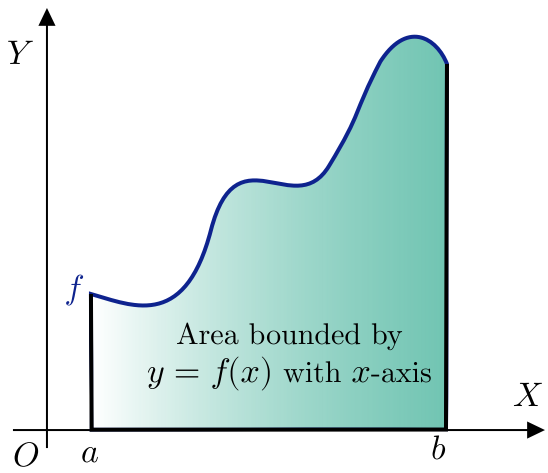


Figure 1. Function f on $[a, b]$

To understand this, we need the following definitions:

Definition 2. Consider a closed interval $I = [a, b]$. By a partition of I we mean a finite set $P = \{x_0, x_1, \ldots, x_n\}$ of points from I such that $a = x_0, x_n = b$ and $x_0 < x_1 < \cdots < x_n$. The interval $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$ are called the subintervals of the interval $I = [a, b]$. By $\Delta_1, \Delta_2, \ldots, \Delta_n$ we denote the length of subintervals $I_1, I_2, \ldots, I_n$ respectively, i.e., $\Delta_i = x_{i-1} - x_i$ for $i = 1, 2, \ldots, n$. It is obvious that, in a partition of subintervals of equal length, as we increase the number of points in the partition P (i.e., the value of n), the length of each subinterval decreases.

Suppose, we divide the interval $I = [a, b]$ into three subintervals (i.e., $n = 3$) and we take the partition $P = \{a = x_0, x_1, x_2, x_3 = b\}$ as shown in the following figure.

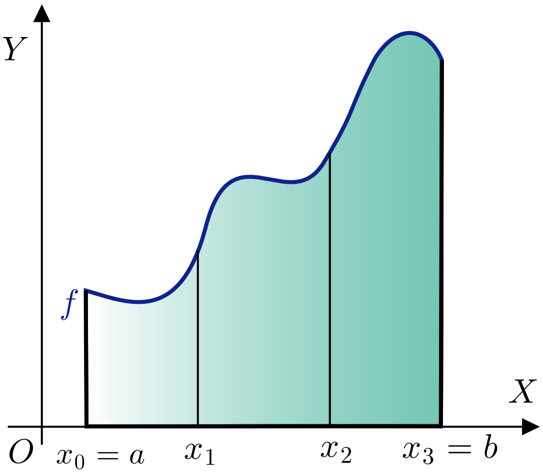


Figure 2. Division of interval into three subintervals

Define $m_i = \inf_{x \in I_i} f(x)$ and $M_i = \sup_{x \in I_i} f(x)$, where $i = 1, 2, 3$. Then, in Figure 3:

$A_1 =$ area of first rectangle bounded between x_0 and $x_1 = M_1(x_1 - x_0) = M_1\Delta_1$;
$A_2 =$ area of next rectangle bounded between x_1 and $x_2 = M_2(x_2 - x_1) = M_2\Delta_2$;
$A_3 =$ area of third rectangle bounded between x_2 and $x_3 = M_3(x_3 - x_2) = M_3\Delta_3$.

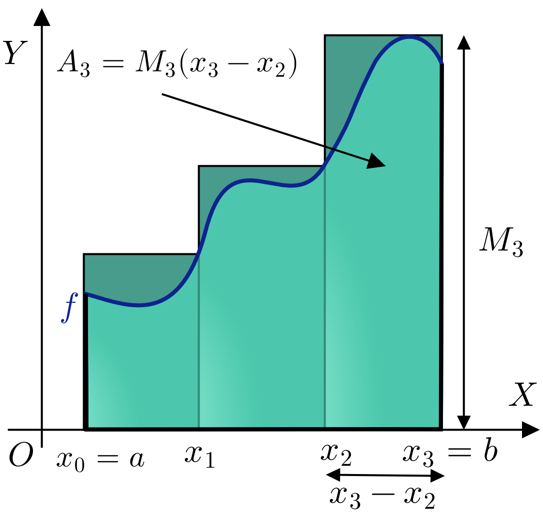


Figure 3. Upper Sum

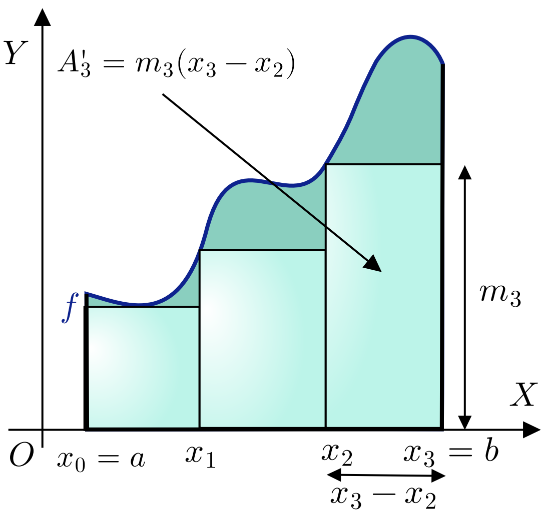


Figure 4. Lower Sum

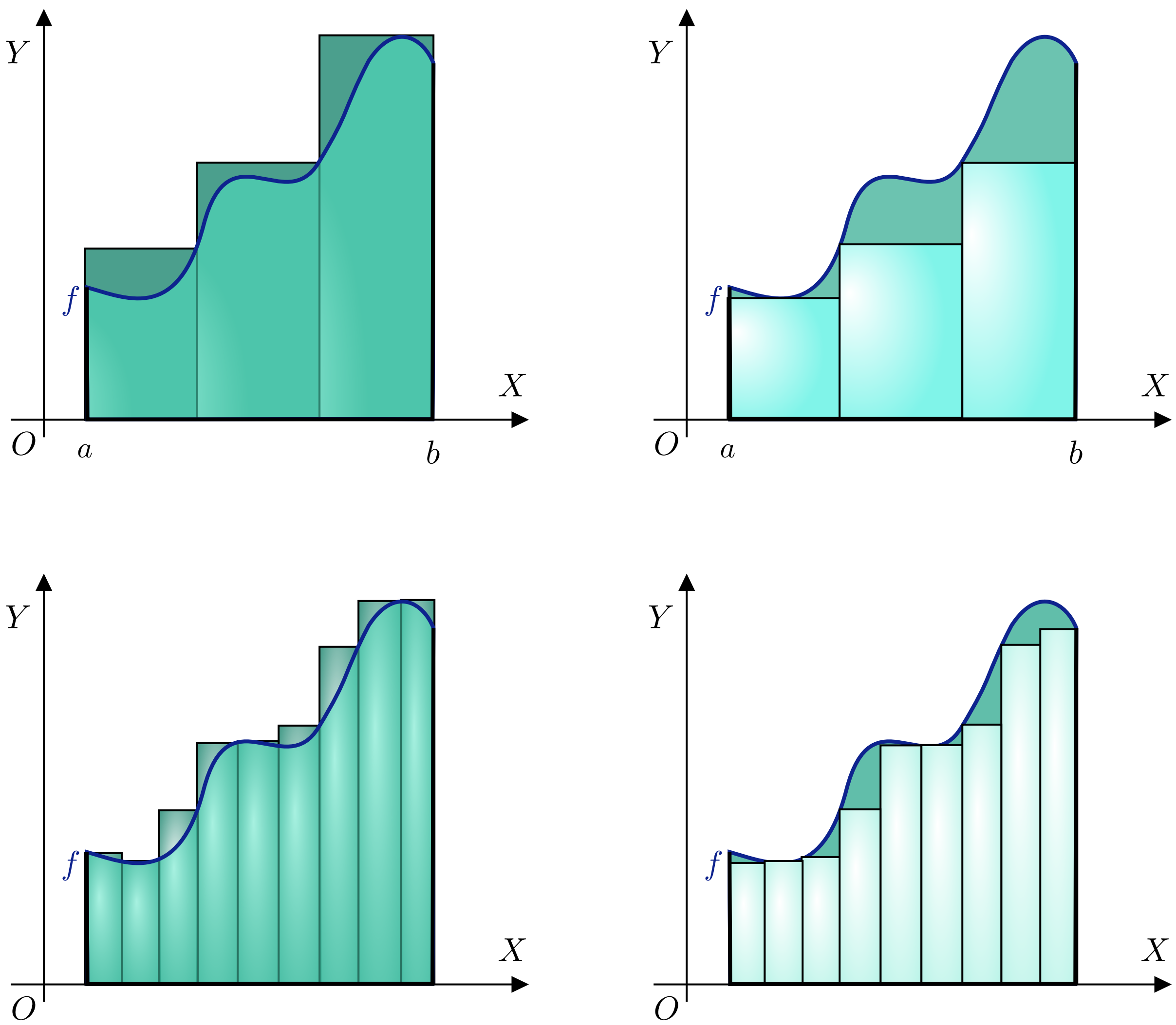
Then, the sum of all these areas is called the Upper Sum of f over the partition P and it is denoted by $U(P, f)$, i.e.:

$$U(P, f) = M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 = \sum_{i=1}^{3} M_i\Delta_i.$$

Similarly, in Figure 4 we define the Lower Sum, denoted by $L(P, f)$ and

$$L(P, f) = m_1\Delta_1 + m_2\Delta_2 + m_3\Delta_3 = \sum_{i=1}^{3} m_i\Delta_i.$$

It is clear that, the value of $U(P, f)$ is some larger than the exact area bounded by the cure $f(x)$ with the x-axis from a to b; and the value of $L(P, f)$ is some smaller than the exact area bounded by the curve $f(x)$ with the x-axis from a to b. Thus, there is an excess of area in $U(P, f)$ and a lack of area in $L(P, f)$. It is obvious that, as we increase the number of points in the partition P, the rectangles becomes more narrower and the length of subintervals decreases. As the rectangles becomes more narrower, the upper sum $U(P, f)$ starts to decreases and so the excess of area in $U(P, f)$ decreases. Similarly, as the rectangles becomes more narrower, the lower sum $L(P, f)$ starts to increase and the lack of area in $L(P, f)$ decreases as well.



Upper Sum decreases

Lower Sum increases

Figure 5.

Finally, as the number of points (we denote it by "n") in the partition P tends to infinite, then the upper sum $U(P, f)$ reaches to a definite value called the *Upper Riemann Integral* and denoted by $\overline{\int_a^b} f(x)\, dx$. Similarly, as the number of points in the partition P tends to infinite, then the lower sum $L(P, f)$ reaches to a definite value called the *Lower Riemann Integral* and denoted by $\underline{\int_a^b} f(x)\,dx$. Thus:

$$\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} M_i \Delta_i \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} m_i \Delta_i.$$

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then the function f is called *Riemann integrable* over $[a, b]$ and the common value of upper and lower Riemann integrals is denoted by $\int_a^b f(x)dx$ and it is equal to the area bounded by the curve with the x-axis from $x = a$ to $x = b$.

Theorem 3 (First Fundamental Theorem of Integral Calculus). *Let f be an integrable function over $[a, x]$ for each $x \in [a, b]$. Then the function F defined by:*

$$F(x) = \int_a^x f(t)dt, \quad x \in (a, b)$$

is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$.

Definition 3 (Antiderivative or Primitive of a function). A function F is called a primitive or antiderivative of a function f on an open interval (a, b) if $F'(x) = f(x)$ for all $x \in (a, b)$.

For example, the function $\sin x$ is a primitive of the function $\cos x$ in every interval. Notice that, the function $\sin x$ as well as the function $\sin x + c$, where c is any arbitrary constant, a primitive of $\cos x$. Therefore, primitive of a function are not unique.

Theorem 4 (Second Fundamental Theorem of Integral Calculus). If F is an anti-derivative (primitive) of a continuous function f on (a, b), then:

$$\int_a^x f(t)dt = F(x) - F(a) \quad \text{for all} \quad x \in [a, b].$$

List of some fundamental integrals

(a) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \neq -1$;

(b) $\displaystyle\int \frac{1}{x} dx = \ln x + c$;

(c) $\displaystyle\int e^{mx} dx = \frac{e^{mx}}{m} + c$, where $m \neq 0$;

(d) $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + c$, where $a > 0$;

(e) $\int \sin x dx = -\cos x + c;$

(f) $\int \cos x dx = \sin x + c;$

(g) $\int \sec^2 x dx = \tan +c;$

(h) $\int \text{cosec}^2 x dx = -\cot +c;$

(i) $\int \sec x \tan x dx = \sec +c;$

(j) $\int \sec x dx = \ln(\sec x + \tan x) + c;$

(k) $\int \text{cosec} x \cot x dx = -\text{cosec} x + c;$

(l) $\int \text{cosec} x dx = \ln(\text{cosec} - \cot x) + c;$

(m) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$, where $|x| < 1$;

(n) $\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1} x + c$ or $-\cot^{-1} x + c;$

(o) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$ or $-\text{cosec}^{-1} x + c$, where $|x| > 1$;

(p) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c;$

(q) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c;$

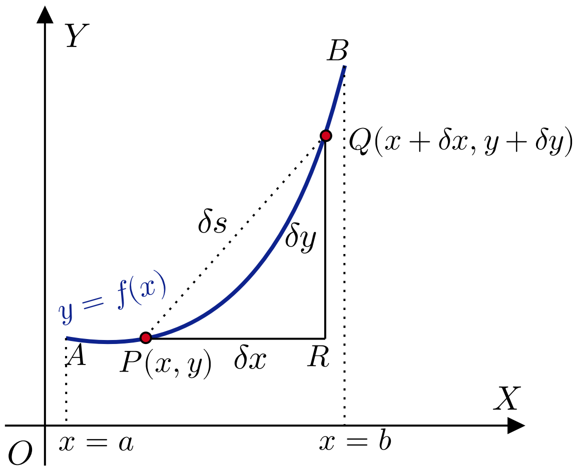
(r) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c;$

(s) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right| + c;$

(t) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c.$

Length of Curves (Rectification)

The finding of the length of a line is quite simple task in geometry. But, in case of an irregular curve given by an equation $y = f(x)$ it is not such an easy task. For this purpose, we use the concept of calculus. Suppose, we have to find the length of arc of the curve given by $y = f(x)$, from point A to the point B, i.e., the arcAB.



Suppose $P(x, y)$ be any point on arcAB, and $Q(x + \delta x, y + \delta y)$ be any point on this arc in the vicinity of point P. Suppose arc$PQ = \delta s$. Now consider the triangle PRQ. Then it is obvious that: $(PQ)^2 = (PR)^2 + (RQ)^2$, i.e., $(PQ)^2 = \delta x^2 + \delta y^2$. As the point Q tends to the point P, i.e., $\delta x, \delta y \to 0$, then arc$PQ \to \delta s$. Therefore, $\delta s^2 = \delta x^2 + \delta y^2$. Since $\delta x, \delta y \to 0$, the small quantities $\delta s, \delta x, \delta y$ now reduce into the infinitely small quantities ds, dx, dy respectively. Thus, we obtain $(ds)^2 = (dx)^2 + (dy)^2$, or:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Now, the whole length of the arc AB can be obtained by summing (integrating) ds from $x = a$ to $x = b$, i.e.:

$$\text{arc}(AB) = \int_{x=a}^{b} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{2}$$

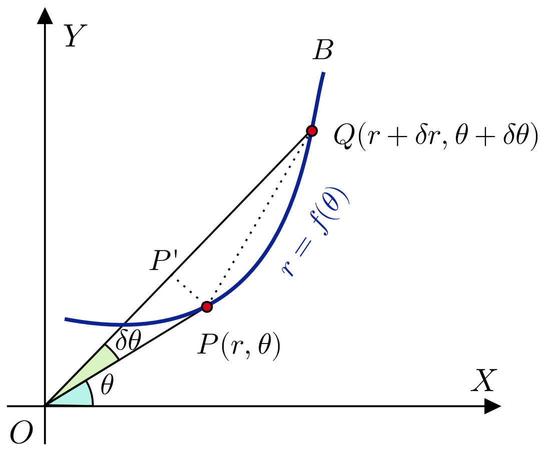
The formula (2) is useful when the integral can be performed easily with respect to x. The following forms can be used as per the convenience and requirement:

(A) **Cartesian form:** $\text{arc}(AB) = \int_{y=c}^{d} ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$

(B) **Parametric form:** $\text{arc}(AB) = \int_{t=t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$

(C) **Polar form:** $\text{arc}(AB) = \int_{\theta=\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$, or:

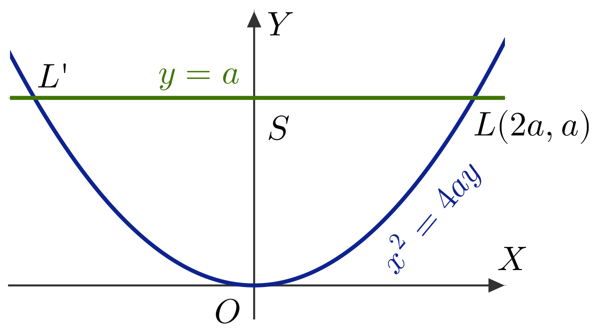
$$\text{arc}(AB) = \int_{r=r_1}^{r_2} ds = \int_{r_1}^{r_2} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2} dr.$$



$\angle POX = \theta,\ \angle QOX = \theta + \delta\theta,\ OP' \approx OP = r,\ OQ = r + \delta r, PP' \approx r\delta\theta,\ \delta s \approx \sqrt{(\delta r)^2 + (r\delta\theta)^2}.$

Example 28. Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.

Solution: Since $x^2 = 4ay$, we have $y = \frac{x^2}{4a}$ and $\frac{dy}{dx} = \frac{x}{2a}$.



The required length of the arc is the arcOL. Now

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{x}{2a}\right)^2}$$
$$= \frac{1}{2a}\sqrt{(2a)^2 + x^2}.$$

Therefore:

$$\begin{aligned}
\text{arc}OL &= \int_{x=0}^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \frac{1}{2a}\int_{x=0}^{2a} \sqrt{(2a)^2+x^2} \\
&= \frac{1}{2a}\left[\frac{x}{2}\sqrt{(2a)^2+x^2} + \frac{(2a)^2}{2}\ln\left(x+\sqrt{(2a)^2+x^2}\right)\right]_0^{2a} \\
&= \frac{1}{2a}\left[a\sqrt{8a^2} + 2a^2\ln\left(2a+\sqrt{8a^2}\right)\right] - \frac{1}{2a}\left[2a^2\ln(2a)\right] \\
&= a\left[\sqrt{2} + \ln\left(2a+\sqrt{8a^2}\right) - \ln(2a)\right] \\
&= a\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right].
\end{aligned}$$

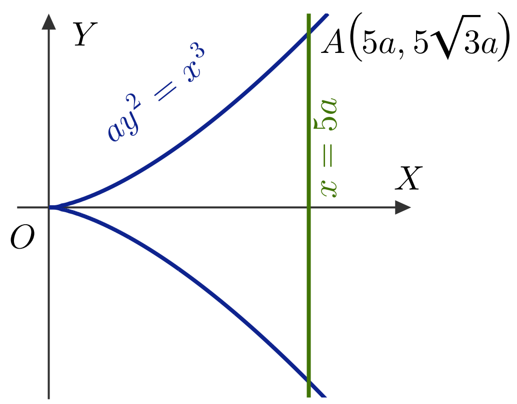
Thus, the required length, $\text{arc}OL = a\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$. □

Example 29. Find the length of arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.

Solution: Given equation of semi-cubical parabola is:

$$ay^2 = x^3.$$

Hence, $y = \frac{x^{3/2}}{\sqrt{a}}$, therefore, $\frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{a}}$. The required length of the arc is the arcOA.



Now:

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{3\sqrt{x}}{2\sqrt{a}}\right)^2} = \frac{1}{2\sqrt{a}}\sqrt{4a+9x}.$$

Therefore:

$$\begin{aligned}
\text{arc}OA &= \int_{x=0}^{5a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \frac{1}{2\sqrt{a}}\int_{x=0}^{5a}\sqrt{4a+9x}\\
&= \frac{1}{2\sqrt{a}}\cdot\frac{2}{3\cdot 9}\left[(4a+9x)^{3/2}\right]_0^{5a}\\
&= \frac{1}{27\sqrt{a}}\left[7^3a^{3/2}-2^3a^{3/2}\right]\\
&= \frac{335}{27}a.
\end{aligned}$$

Thus, the required length, $\text{arc}OB = \dfrac{335}{27}a.$ □

Example 30. Find the whole length of:

(a) cardioid: $r = a(1+\cos\theta)$;

(b) cycloid: $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$;

(c) astroid: $x = a\cos^3 t,\ y = a\sin^3 t$ (or, $x^{2/3}+y^{2/3} = a^{2/3}$);

(d) circle: $x = a\cos\theta,\ y = a\sin\theta$ (or, $x^2+y^2 = a^2$);

Solution: (a) Given equation of cardioid is

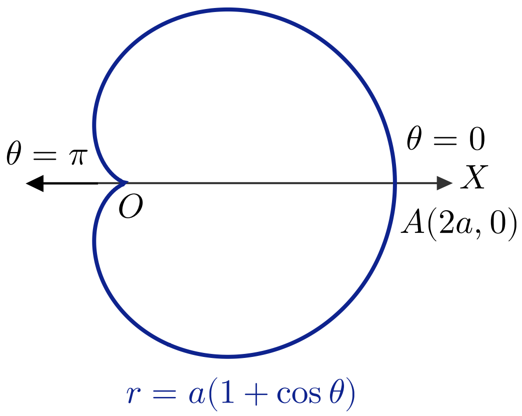
$$r = a(1+\cos\theta).$$

Therefore, $\dfrac{dr}{d\theta} = -a\sin\theta$ (polar form). Hence:

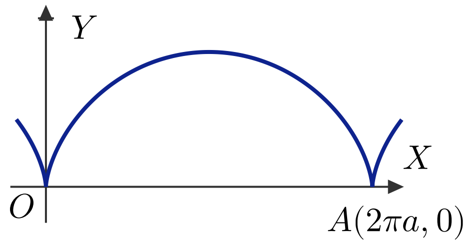
$$\begin{aligned}
\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} &= \sqrt{a^2(1+\cos\theta)^2+(-a\sin\theta)^2}\\
&= 2a\cos(\theta/2).
\end{aligned}$$

Now, the whole length of the Cardioid

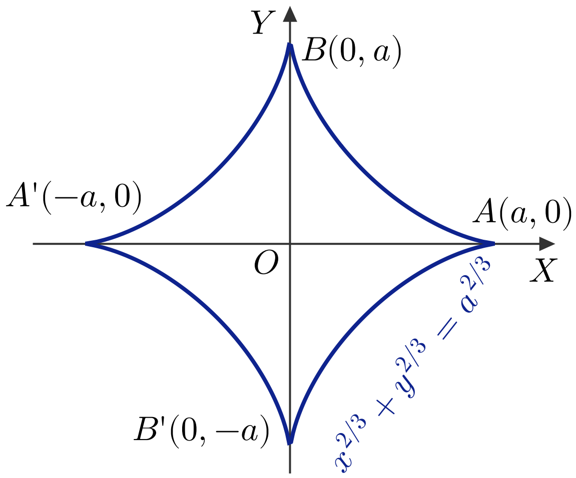
$$\begin{aligned}
L &= 2\int_{-\pi}^{0}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta\\
&= 2\int_{-\pi}^{0} 2a\cos(\theta/2)\,d\theta\\
&= 4a\left[2\sin(\theta/2)\right]_{-\pi}^{0}\\
&= 8a.
\end{aligned}$$



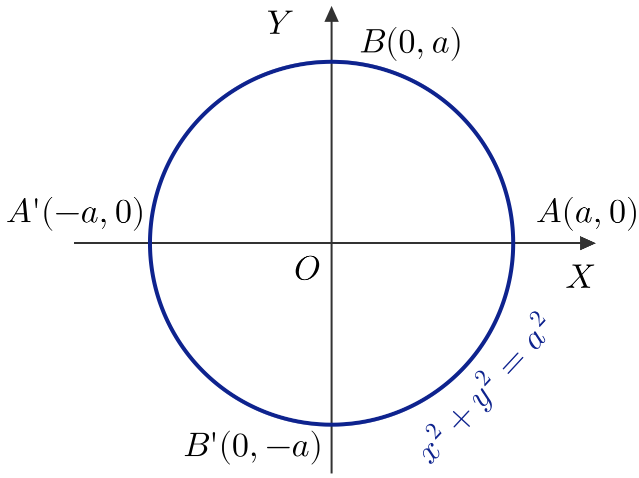
(a) Cardioid



(b) Cycloid



(c) Astroid



(d) Circle

(b) Here, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ (parametric form), so:

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \frac{dy}{d\theta} = a\sin\theta.$$

Therefore:

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{[a(1-\cos\theta)]^2 + (a\sin\theta)^2} = 2a\sin(\theta/2).$$

Now, the whole length of the cycloid is the arcOA. Note that, at point O, $\theta = 0$ and at point A, $\theta = 2\pi$. Therefore,

$$\begin{aligned} OA &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^{2\pi} 2a\sin(\theta/2) d\theta \\ &= 2a\left[-2\cos(\theta/2)\right]_0^{2\pi} \\ &= 8a. \end{aligned}$$

(c) Here, $x = a\cos^3 t,\ y = a\sin^3 t$ (parametric form), so:

$$\frac{dx}{dt} = -3a\cos^2 t\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

Therefore:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{[-3a\cos^2 t\sin t]^2 + \left(3a\sin^2 t\cos t\right)^2} = 3a\sin t\cos t.$$

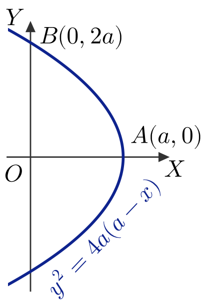
Now, the whole length of the astroid is the $L =$arc$ABA'B'A$. Note that, at point A, $t = 0$ and at point B, $\theta = \pi/2$. Therefore,

$$\begin{aligned} L &= 4 \times \text{arc}AB = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 4\int_0^{\pi/2} 3a\sin t\cos t dt \\ &= 6a\int_0^{\pi/2} \sin(2t)dt = 3a\left[-\cos(2t)\right]_0^{\pi/2} \\ &= 6a. \end{aligned}$$

(d) Try yourself. □

Exercise (Assignment)

(Q.1) Find the length of the arc of the parabola $y^2 = 4a(a - x)$ cut off by the y-axis.



Ans: arc BA$= a[\sqrt{2} + \ln(1 + \sqrt{2})]$.

(Q.2) By finding the length of of the curve show that the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ is divided in the ratio 1 : 3 at $\theta = \frac{2\pi}{3}$.

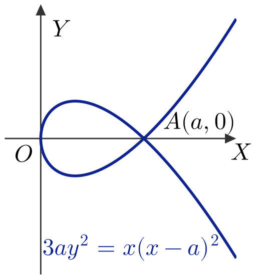
(Q.3) Find the length of the curve $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/3$.

Ans: $\ln\left(2 + \sqrt{3}\right)$.

(Q.4) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.

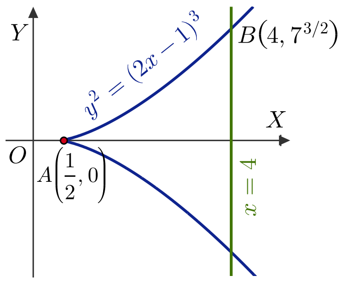
Ans: $a\,(15/16 + \ln 2)$.

(Q.5) Find the whole length of the loop of the curve $3ay^2 = x(x-a)^2$



Ans: $4a/\sqrt{3}$.

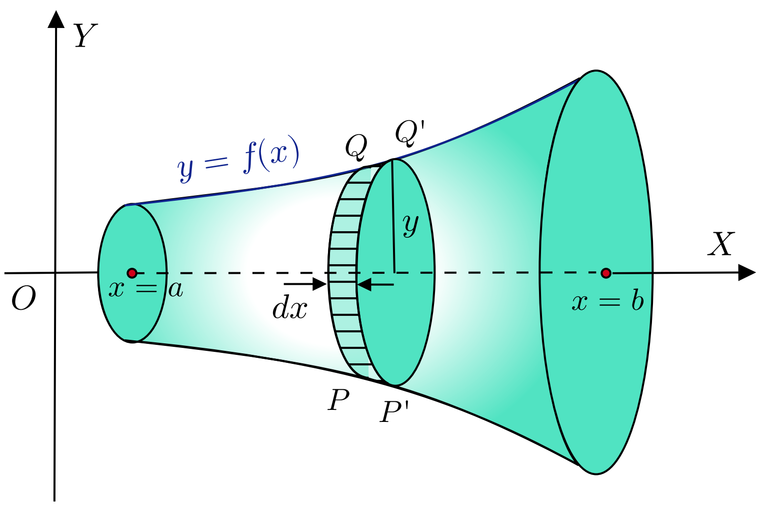
(Q.6) Find the length of the curve $y^2 = (2x-1)^3$ cut off by the line $x = 4$.



Ans. 37.85

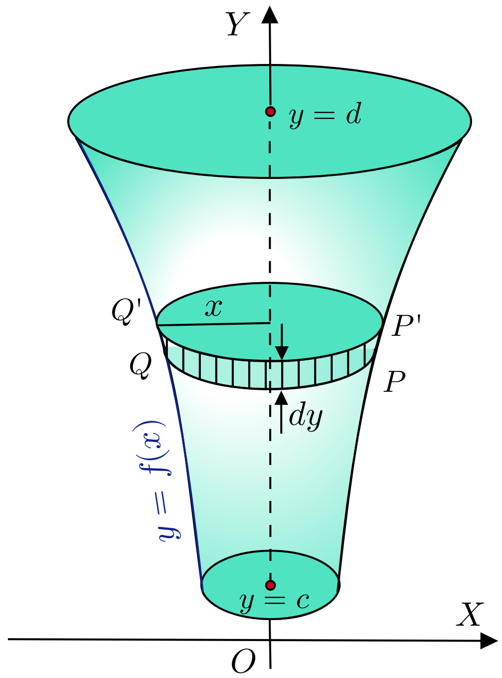
Volume of solid generated by the revolution of area of curves

The idea. Suppose, the area enclosed by an arc of the curve $y = f(x)$ from $x = a$ to $x = b$ with x-axis is revolved about the x-axis. Then a solid shape is thus generated and we have to find the volume of this solid. For this, we cut vertically this solid into a large number (say "n") of thin discs each of thickness δx. Consider such a disc $PP'QQ'$ shown in the figure below. Then, the volume of the solid can be obtained by adding the volume of all such discs.

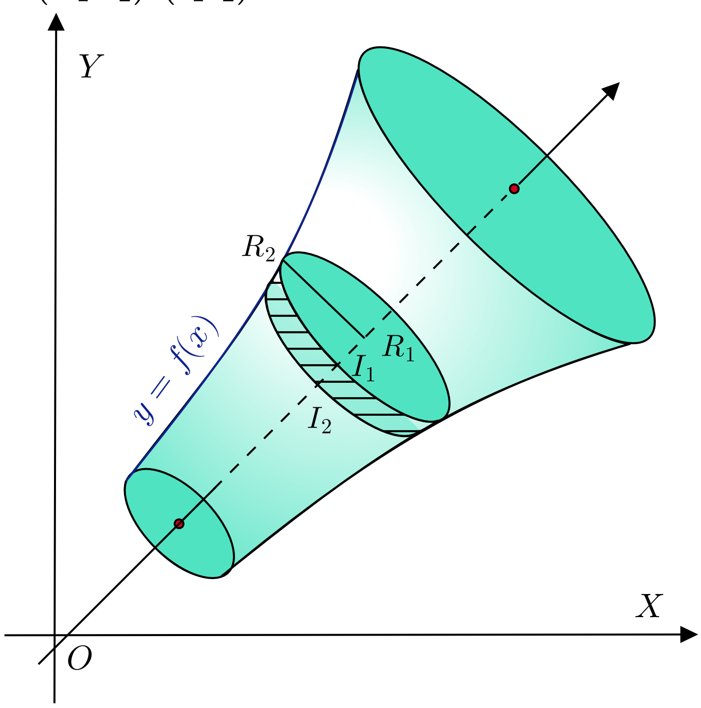


Now we obtained the volume of disc $PP'QQ'$. Let the coordinate of point Q is (x, y), then the volume of the disc $PP'QQ'$ will be $\delta v = \pi y^2 \delta x$. As $n \to \infty$ the small quantities δx and δv reduce into the infinitely small quantities dx and dv respectively. Therefore, the volume of disc $PP'QQ'$ is $dv = \pi y^2 dx$. Thus, the volume of the solid generated is $V = \int_{x=a}^{b} dv = \int_a^b \pi y^2 dx$.

Remark 1. (A) If the area bounded by the arc of curve (from $y = c$ to $y = d$) with y-axis is revolved about the y-axis, then the volume of generated solid is given by $V = \int_{y=c}^{d} dv = \int_{c}^{d} \pi x^2 dy$.



(B) If the area bounded by the arc of curve (from point A to B) with line L is revolved about the line L, then the volume of generated solid is given by $V = \int_{A}^{B} dv = \int_{A}^{B} \pi (R_1 R_2)^2 (I_1 I_2)$.

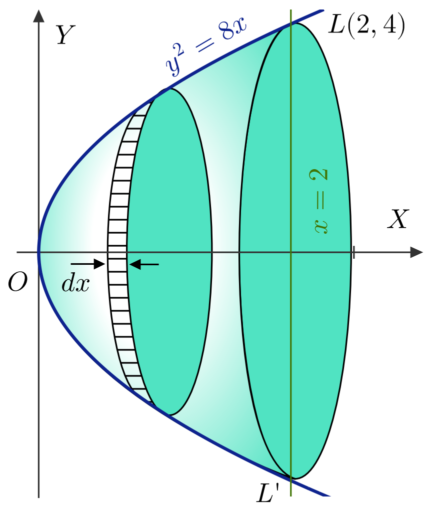


(C) In polar form the volume of revolution about the initial line is:

$$V = \frac{2}{3} \int_{\theta=\theta_1}^{\theta_2} \pi r^3 \sin\theta d\theta.$$

Example 31. Find the volume generated by revolving the area in the first quadrant bounded by the parabola $y^2 = 8x$ and its latus rectum about the x-axis.

Solution: Given equation of parabola is $y^2 = 8x$ and its latus rectum is the line LL' whose equation is $x = 2$, as shown in the figure. The required volume is the volume generated by the area bounded by the arc OL from $x = 0$ to $x = 2$ with the x-axis. Therefore, the required volume is:

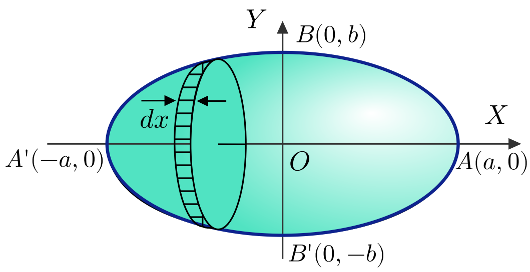


$$V = \int_0^2 \pi y^2 dx = \int_0^2 \pi(8x)dx = 16\pi.$$

Thus the required volume $V = 16\pi$. □

Example 32. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, about the x-axis.

Solution: Given equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The ellipse is shown the figure. The required volume is the volume generated by the area bounded by the arc $A'BA$ from $x = -a$ to $x = a$ with the x-axis. Therefore, the required volume is:

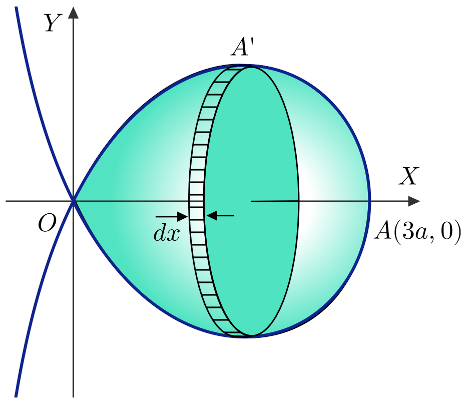


$$\begin{aligned} V &= \int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi \frac{b^2}{a^2}(a^2 - x^2)dx \\ &= \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2)dx = \frac{4\pi}{3}ab^2. \end{aligned}$$

Thus the required volume $V = \frac{4\pi}{3}ab^2$. □

Example 33. The curve $y^2(a + x) = x^2(3a - x)$ revolves about the axis of x. Find the volume generated by the loop.

Solution: Given equation of curve is $y^2(a+x) = x^2(3a-x)$. Given curve is symmetric about the x-axis and cuts the x-axis at points $O(0,0)$ and $A(3a,0)$, hence a loop $OA'AO$ is formed. The curve is shown the figure. The required volume is the volume generated by the area bounded by the arc $OA'A$ from $x=0$ to $x=3a$ with the x-axis. Therefore, the required volume is:

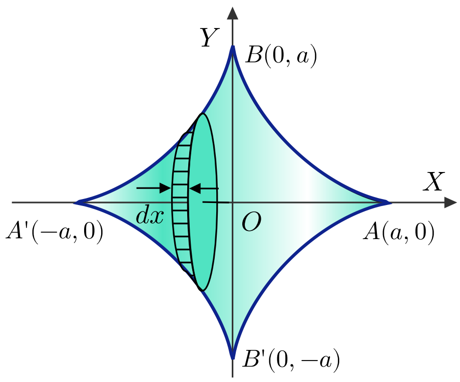


$$\begin{aligned}
V &= \int_0^{3a} \pi y^2 dx = \int_0^{3a} \pi \frac{x^2(3a-x)}{a+x} dx \\
&= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a}\right] dx \\
&= \pi \left[-\frac{x^3}{3} + 2ax^2 + 4a^3 \ln(x+a)\right]_0^{3a} \\
&= \pi a^3 [8\ln(2) - 3].
\end{aligned}$$

Thus the required volume $V = \pi a^3[8\ln(2) - 3]$. □

Example 34. Find the volume of the solid generated by the revolution of the curve: $x = a\cos^3 t,\ y = b\sin^3 t$, or $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, about the x-axis.

Solution: Given equation of curve (astroid) is $x = a\cos^3 t,\ y = b\sin^3 t$. The curve is shown in the figure. The volume generated by revolving the area bounded by the arc $A'BA$ with x-axis from $x=-a$ to $x=a$ about the x-axis. From the figure it is clear that this volume is equal to the twice of the volume generated by the revolution area bounded by the arc BA with the x-axis from $x=0$ to $x=a$ about the x-axis.



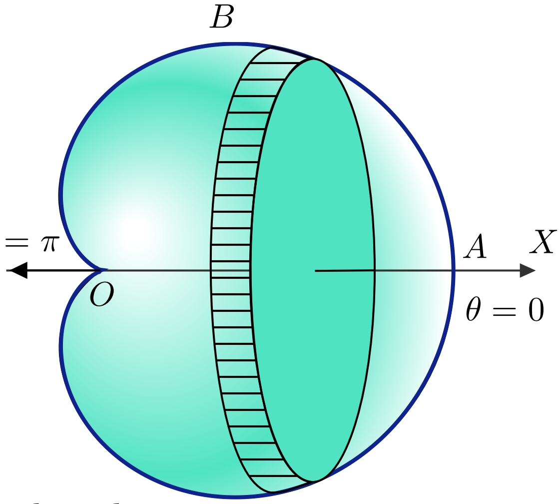
At point $x=0$ we have $t=\frac{\pi}{2}$ and at $x=a$ we have $t=0$. Thus, the required volume:

$$\begin{aligned}
V &= 2\int_{t=\pi/2}^{0} \pi y^2 dx = 2\int_{\pi/2}^{0} \pi (b\sin^3 t)^2 d(a\cos^3 t) \\
&= 6\pi ab^2 \int_0^{\pi/2} \sin^7 t \cos^2 t dt = 6\pi ab^2 \frac{\Gamma(4)\Gamma(\frac{3}{2})}{2\Gamma(\frac{11}{2})} \\
&= 6\pi ab^2 \frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot 3!}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = 32\pi\frac{ab^2}{105}.
\end{aligned}$$

Thus the required volume $V = 32\pi\dfrac{ab^2}{105}$. □

Example 35. Find the volume of the solid generated by the revolution of the cardioid: $r = a(1+\cos\theta)$ about the initial line from $\theta = 0$ to $\theta = \pi$.

Solution: Given equation of cardioid is $r = a(1+\cos\theta)$. The cardioid is shown in the figure. It is clear from the figure that the required volume is formed by the area bounded by the arc OBA from $\theta = 0$ to $\theta = \pi$ with the initial line. Thus, the required volume:



$$\begin{aligned} V &= \frac{2}{3}\int_{\theta=0}^{\pi} \pi r^3 \sin\theta d\theta \\ &= \frac{2\pi}{3}\int_{\theta=0}^{\pi} \pi a^3(1+\cos\theta)^3 \sin\theta d\theta. \end{aligned}$$

Putting $1+\cos\theta = t$ we obtain $\sin\theta d\theta = -dt$ and

$$\begin{aligned} \theta \to 0 &\implies t \to 2 \\ \theta \to \pi &\implies t \to 0. \end{aligned}$$

Hence:

$$V = \frac{2\pi a^3}{3}\int_0^2 t^3 dt = \frac{2\pi a^3}{3}\left[\frac{t^3}{3}\right]_0^2 = \frac{8\pi a^3}{3}.$$

Thus the required volume $V = \dfrac{8\pi a^3}{3}$. □

Exercise (Assignment)

(Q.1) Find the volume of the spindle-shaped solid generated by revolving the area of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

Ans. $\dfrac{32\pi a^3}{105}$.

(Q.2) Find the volume of sphere of radius a.

Hint: revolve the area of upper half of the circle $x^2 + y^2 = a^2$ about the x-axis.

(Q.3) Find the volume of the solid generated by the revolution of area of parabola $y^2 = 4ax$ formed by the its arc from $x = 0$ to $x = h$ and x-axis about the x-axis.

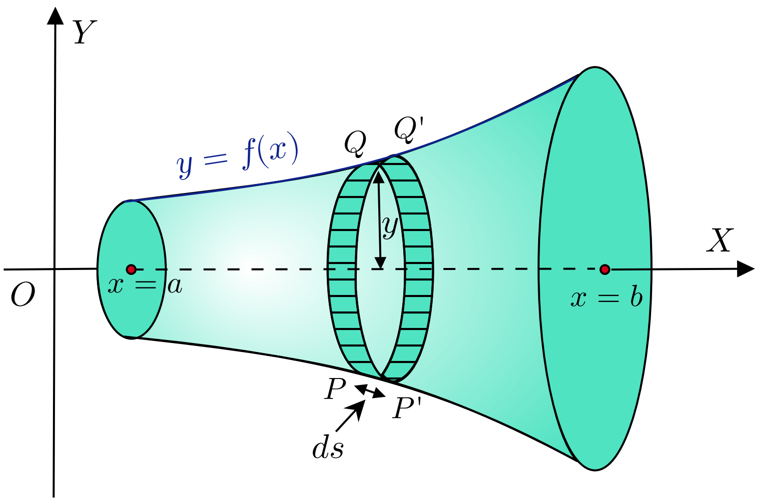
Ans: $2a\pi h^2$.

(Q.4) Prove that the volume of a right circular cone of height h and base of radius r is $\frac{1}{3}\pi r^2 h$.

Hint: It is generated by the revolution of the line $y = \frac{h}{r}(r-x)$ about y-axis.

Surface of revolution

The idea. Suppose, an arc of the curve $y = f(x)$ from $x = a$ to $x = b$ is revolved about x-axis. Then a solid shape is thus generated and we have to find the surface area of this solid. For this, we cut vertically the surface of solid into a large number (say "n") of thin rings each of thickness ds. Consider such a ring $PP'QQ'$ shown in the figure below. Then, the surface area of the solid can be obtained by adding the surface area of all such rings.



Now we obtained the surface area of the ring $PP'QQ'$. Let the coordinate of point Q is (x, y), then the surface area of the ring $PP'QQ'$ will be $\delta S = 2\pi y \delta s$, where δs is the length of the arc QQ'. As $n \to \infty$ the small quantities δs and δS reduce into the infinitely small quantities ds and dS respectively. Therefore, the surface area of the ring $PP'QQ'$ is $dS = 2\pi y ds$.

We know that the length of the arc $QQ' = ds = \sqrt{(dx)^2 + (dy)^2}$. Therefore, the surface area of the small ring:

$$dS = 2\pi y ds = 2\pi y\sqrt{(dx)^2 + (dy)^2} = 2\pi y\sqrt{\left(1 + \frac{dy}{dx}\right)^2}dx.$$

Now the surface area of the whole revolution can be obtained by integrating dS from $x = a$ to $x = b$, i.e., the required surface:

$$S = \int_{x=a}^{b} ds = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx.$$

Remark 2. (A) If the curve is revolved about the y-axis, then the surface of revolution:

$$S = \int_{y=c}^{d} ds = \int_{c}^{d} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

(B) If the curve is revolving about the x-axis or the initial line (polar form), then the surface of revolution:

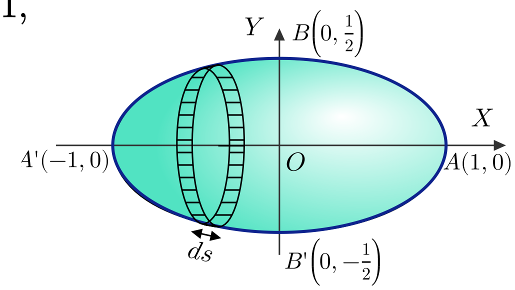
$$S = \int_{\theta=\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

(C) If the curve is revolving about the y-axis or the line $\theta = \pi/2$ (polar form), then the surface of revolution:

$$S = \int_{\theta=\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 36. Find the surface area of the solid generated by the revolution of ellipse $x^2 + 4y^2 = 1$ about x-axis.

Solution. The equation of ellipse is $x^2 + 4y^2 = 1$, i.e., $\frac{dy}{dx} = -\frac{x}{4y}$. Therefore:



$$\begin{aligned}
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \frac{1}{4y}\sqrt{x^2 + 16y^2} \\
&= \frac{1}{4y}\sqrt{4 - 3x^2}.
\end{aligned}$$

Now, the surface area of the revolution of ellipse is equal to the twice the area of revolution of the arc(BA) about x-axis. On this arc, the value of x varies from $B(x = 0)$ to $A(x = 1)$. Thus, the required area:

$$\begin{aligned}
S &= 2\int_{x=0}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dx \\
&= 4\pi \int_0^1 y \cdot \frac{1}{4y}\sqrt{4 - 3x^2}\, dx = \pi\sqrt{3}\int_0^1 \sqrt{\frac{4}{3} - x^2}\, dx \\
&= \pi\sqrt{3}\left[\frac{x}{2}\sqrt{\frac{4}{3} - x^2} + \frac{4}{2\cdot 3}\sin^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_0^1 \\
&= \pi\left[\frac{1}{2} + \frac{\pi}{3\sqrt{3}}\right].
\end{aligned}$$

Thus, the required surface area $S = \pi\left[\frac{1}{2} + \frac{\pi}{3\sqrt{3}}\right]$. □

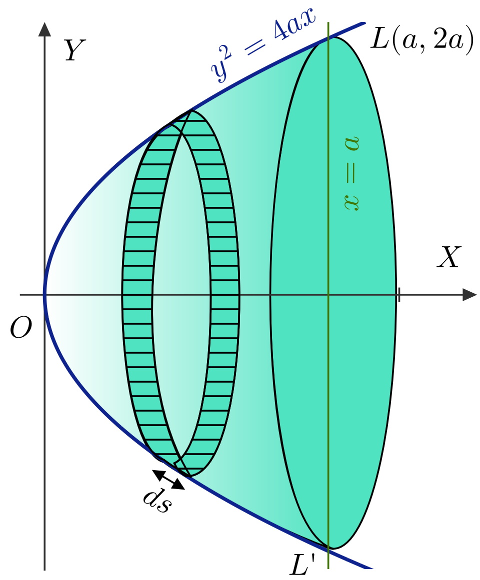
Example 37. Find the surface area of the solid generated by the revolution of the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.

Solution. Given equation of parabola is:

$$y^2 = 4ax.$$

Hence, $\frac{dy}{dx} = \frac{2a}{y}$. The equation of latus rectum LL' of parabola is $x = a$, and the required surface of solid is the surface of solid generated by the revolution of arc OL about x-axis. Now:

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x+a}{x}}.$$



The value of x varies from $O(x = 0)$ to $L(x = a)$. Thus, the required area:

$$\begin{aligned} S &= \int_{x=0}^{a} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_{x=0}^{a} \sqrt{4ax}\sqrt{\frac{x+a}{x}}\,dx \\ &= 4\pi\sqrt{a}\int_{x=0}^{a} \sqrt{x+a}\,dx \\ &= \frac{8\pi a^2}{3}[2\sqrt{2} - 1]. \end{aligned}$$

Thus, the required surface area $S = \frac{8\pi a^2}{3}[2\sqrt{2} - 1]$. □

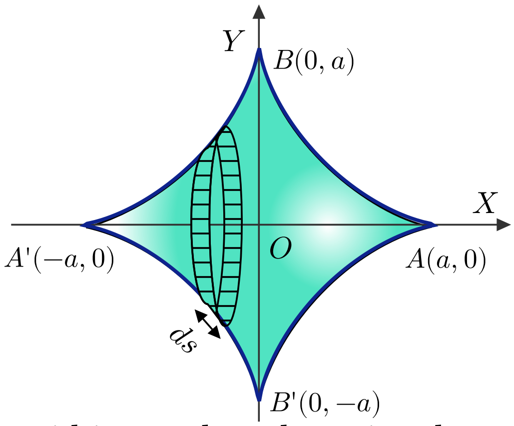
Example 38. Find the surface area of the solid generated by the revolution of astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

Solution. The parametric equation of astroid is:

$$x = a\cos^3 t,\ y = a\sin^3 t.$$

Therefore,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3a\sin t\cos t.$$



Now, the surface area of the revolution of astroid is equal to the twice the area of revolution of the Acr(BA) about x-axis. On this arc, the value of t varies from $B(t=0)$ to $A(t=\pi/2)$. Thus, the required area:

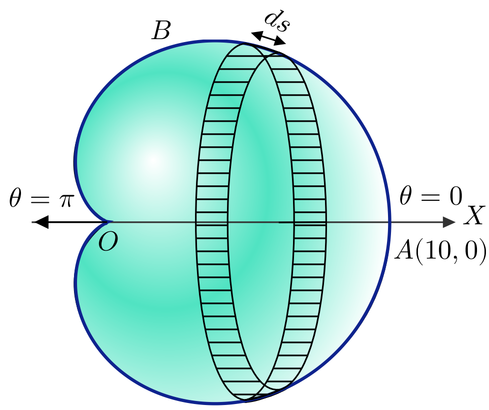
$$\begin{aligned}
S &= 2\int_{t=0}^{\pi/2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 4\pi\int_0^{\pi/2} (a\sin^3 t\cdot 3a\sin t\cos t)\ dt = 12\pi a^2\int_0^{\pi/2} \sin^4 t\cos t\ dt \\
&= 12\pi a^2\frac{\Gamma(\frac{5}{2})\Gamma(1)}{2\Gamma(\frac{7}{2})} \\
&= \frac{12\pi a^2}{5}.
\end{aligned}$$

Thus, the required surface area $S = \dfrac{12\pi a^2}{5}$. □

Example 39. Find the surface area of the solid generated by the revolution of cardioid $r = 5(1+\cos\theta)$ about the initial line.

Solution. The equation of cardioid is $r = 5(1+\cos\theta)$, i.e., $\dfrac{dr}{d\theta} = -5\sin\theta$. Therefore,

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 5\sqrt{2+2\cos\theta} = 10\cos(\theta/2).$$



Now, the surface area of the revolution of cardioid is equal to the area of revolution of the Acr(OA) about x-axis. On this arc, the value of θ varies from

$A(\theta = 0)$ to $O(\theta = \pi)$. Thus, the required area:

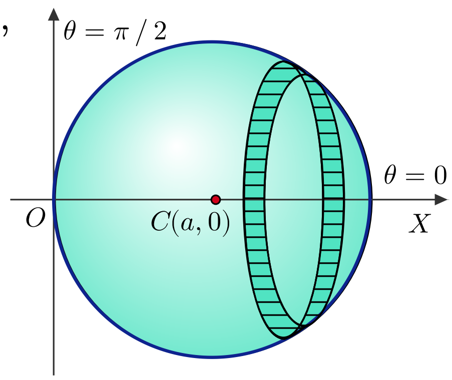
$$\begin{aligned} S &= \int_{\theta=0}^{\pi} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= 20\pi \int_0^{\pi} r \sin\theta \cos(\theta/2)\ d\theta \\ &= 40\pi \int_0^{\pi} 5(1+\cos\theta)\sin\theta\cos(\theta/2)\ d\theta \\ &= 400\pi \int_0^{\pi} \sin(\theta/2)\cos^4(\theta/2)\ d\theta \\ &= 160\pi. \end{aligned}$$

Thus, the required surface area $S = 160\pi$. □

Example 40. Find the surface area of the solid generated by the revolution of circle $r = 2a\cos\theta$ about the initial line.

Solution. The equation of circle is $r = 2a\cos\theta$, i.e., $\dfrac{dr}{d\theta} = -2a\sin\theta$. Therefore:

$$\begin{aligned} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta} \\ &= 2a. \end{aligned}$$



Now, the surface area of the revolution of cardioid is equal to the area of revolution of the Acr(OA) about x-axis. On this arc, the value of θ varies from $A(\theta = 0)$ to $O(\theta = \pi/2)$. Thus, the required area:

$$\begin{aligned} S &= \int_{\theta=0}^{\pi/2} 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= 2\pi \int_0^{\pi/2} r\sin\theta 2a\ d\theta \\ &= 4\pi a \int_0^{\pi/2} 2a\cos\theta\sin\theta\ d\theta \\ &= 8\pi a^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \\ &= 4\pi a^2. \end{aligned}$$

Home Work (Assignment)

(Q.1) Find the area of the surface generated by the revolution of the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t)$ about x-axis.

Ans: $\dfrac{64\pi a^2}{3}$.

(Q.2) Find the surface area of a right circular cylinder of radius r and hight h.

Hint: Right circular cylinder is generated by the revolution of line $y = r$ about x-axis, from $x = 0$ to $x = h$.

(Q.3) Find the surface area of a cone of hight h and radius r.

Hint: Cone is generated by the revolution of line $x = \dfrac{r}{h}(h-y)$ about y-axis, from $y = 0$ to $y = h$).

♣ ♣ ♣ ♣

Unit-IV

Evaluation of integrals using gamma function. Multiple integral: double integral, area by double integral. Evaluation of triple integrals.

Beta and gamma functions

Beta function. For $n, m > 0$ the beta function of n, m is denoted by $\beta(n, m)$ and

$$\beta(n, m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx.$$

For $n > 0$, the gamma function of n is denoted by $\Gamma(n)$ and it is defined by:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx.$$

Property I. Prove that $\beta(n, m) = \beta(m, n)$. (the beta function is symmetric in its arguments).

Proof. By the definition we have:

$$\beta(n, m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx.$$

On putting $1 - x = y$, i.e., $dx = -dy$ we have:

$$\begin{aligned} x \to 0 &\implies y \to 1 \\ x \to 1 &\implies y \to 0. \end{aligned}$$

Hence:

$$\beta(n, m) = -\int_1^0 (1-y)^{n-1}y^{m-1}dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy.$$

As, in definite integral variables are dummy, hence we have:

$$\beta(n, m) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m, n).$$

This proves the result. □

Property II. Prove that $\beta(n, m) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta.$

Proof. By the definition we have:

$$\beta(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx.$$

On putting $x = \sin^2\theta$, i.e., $dx = 2\sin\theta\cos\theta d\theta$ we have:

$$\begin{aligned} x \to 0 &\implies \theta \to 1 \\ x \to 1 &\implies \theta \to \frac{\pi}{2}. \end{aligned}$$

Hence:

$$\begin{aligned} \beta(n,m) &= \int_0^{\pi/2} (\sin^2\theta)^{n-1}\left(1-\sin^2\theta\right)^{m-1}(2\sin\theta\cos\theta d\theta) \\ &= 2\int_0^{\pi/2} \sin^{2n-2}\theta\cos^{2m-2}\theta \cdot \sin\theta\cos\theta d\theta \\ &= 2\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m-1}\theta d\theta. \end{aligned}$$

This proves the required result. □

Property III. Prove that $\Gamma(1) = 1$.

Proof. By the definition we have:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx.$$

On putting $n = 1$ we have:

$$\begin{aligned} \Gamma(1) &= \int_0^\infty e^{-x}x^{1-1}dx \\ &= \int_0^\infty e^{-x}dx \\ &= \left[-e^{-x}\right]_0^\infty = -0 + 1 \\ &= 1. \end{aligned}$$

This proves the required result. □

Property IV. Prove that $\Gamma(n+1) = n\Gamma(n)$ (reduction formula for $\Gamma(n)$). Hence, prove that $\Gamma(n+1) = n!$ if n is a positive integer.

Proof. By the definition we have:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

On replacing n by $n+1$ we have:

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty e^{-x} x^{n+1-1} dx = \int_0^\infty e^{-x} x^n dx \\ &= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n \cdot x^{n-1} \left(-e^{-x}\right) dx \\ &= -0 + 0 + n \int_0^\infty x^{n-1} e^{-x} dx \\ &= n\Gamma(n).\end{aligned}$$

If n is positive integer, then replacing n by $n-1, n-2, \ldots, 2, 1$ in the above formula, we get:

$$\Gamma(n) = (n-1)\Gamma(n-1), \Gamma(n-1) = (n-2)\Gamma(n-2), \ldots, \Gamma(2) = 1\Gamma(1) = 1.$$

On combining all the above results, we get:

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n \cdot (n-1)\Gamma(n-1) = n \cdot (n-1) \cdot (n-2)\Gamma(n-2) \\ &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \\ &= n!\end{aligned}$$

This proves the required result. □

Property V. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof. By the definition we have:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

On putting $n = \frac{1}{2}$ we get:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{1/2-1} dx = \int_0^\infty e^{-x} x^{-1/2} dx.$$

Putting $x = y^2$, i.e., $dx = 2y dy$ we have:

$$\begin{aligned} x \to 0 &\implies y \to 0 \\ x \to \infty &\implies y \to \infty.\end{aligned}$$

Hence:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} y^{-1} \cdot (2y dy) = 2\int_0^\infty e^{-y^2} dy.$$

Since variables are dummy we can write:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy, \text{ and } \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx.$$

On multiplying these two values we get:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

Since $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \frac{\pi}{4}$ (we will prove it later), hence we get:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \cdot \frac{\pi}{4} = \pi.$$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. □

Relation between beta and gamma functions

Theorem 5. *If n, m are positive integers, then prove that:*

$$\beta(n, m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}.$$

Proof. By definition we know that

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

On putting $x = y^2$, i.e., $dx = 2y dy$ we have:

$$\begin{aligned} x \to 0 &\implies y \to 0 \\ x \to \infty &\implies y \to \infty. \end{aligned}$$

Hence, $\Gamma(n) = \int_0^\infty e^{-y^2} y^{2n-2} (2y dy)$, or:

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy. \tag{3}$$

Since variables are dummy we can write:

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx, \text{ and } \Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1} dy.$$

On multiplying these two, we get:

$$\begin{aligned}\Gamma(n)\Gamma(m) &= \left[2\int_0^\infty e^{-x^2} x^{2n-1} dx\right]\cdot\left[2\int_0^\infty e^{-y^2} y^{2m-1} dy\right]\\ &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dxdy.\end{aligned}$$

To evaluate the above double integral, we change the variables into polar coordinates. Then we know by the relation between cartesian and polar coordinates that

$$x = r\cos\theta, y = r\sin\theta, r^2 = x^2+y^2, dxdy = rdrd\theta.$$

Since the limits of integration are from $x, y \to 0$ to $x, y \to \infty$, hence the region of integration is the positive quadrant of xy-plane. We know that in the positive quadrant, the polar coordinates changes from $r \to 0$ to $r \to \infty$ and $\theta \to 0$ to $\theta \to \pi/2$. Hence:

$$\begin{aligned}\Gamma(n)\Gamma(m) &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1} rdrd\theta\\ &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r^{2(n+m)-1}\cos^{2n-1}\theta\sin^{2n-1}\theta drd\theta\\ &= 2\int_0^{\pi/2}\left[2\int_0^\infty e^{-r^2} r^{2(n+m)-1} dr\right]\cos^{2n-1}\theta\sin^{2n-1}\theta d\theta.\end{aligned}$$

From (3) we get $\Gamma(n+m) = 2\int_0^\infty e^{-r^2} r^{2(n+m)-1} dr$. On putting this value in the above equation we get:

$$\begin{aligned}\Gamma(n)\Gamma(m) &= 2\int_0^{\pi/2}\Gamma(n+m)\cos^{2n-1}\theta\sin^{2n-1}\theta d\theta\\ &= 2\Gamma(n+m)\int_0^{\pi/2}\cos^{2n-1}\theta\sin^{2n-1}\theta d\theta.\end{aligned}$$

Also, we know that $\beta(n,m) = 2\int_0^{\pi/2}\cos^{2n-1}\theta\sin^{2n-1}\theta d\theta$. On putting this value in the above equation we get:

$$\Gamma(n)\Gamma(m) = \Gamma(n+m)\beta(n,m).$$

Thus, $\beta(n,m) = \dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$. □

Corollary 6. *If n, m are positive integers, then prove that:*

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Hence, show that $\int_0^{\pi/2} \sin^p\theta d\theta = \int_0^{\pi/2}\cos^p\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}.$

Proof. We know that if m, n are two positive integers then

$$\beta(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}.$$

Also, since $\beta(n,m) = 2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta d\theta$, hence we obtain:

$$\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta d\theta = \frac{\Gamma(n)\Gamma(m)}{2\Gamma(n+m)}.$$

Putting $n = \frac{p+1}{2}, m = \frac{q+1}{2}$, or $p = 2n-1, q = 2m-1$ in the above equation we get:

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Putting $p = 0$ and $q = 0$ respectively in the above relation we get:

$$\int_0^{\pi/2}\sin^p\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}$$

$$\int_0^{\pi/2}\cos^q\theta d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{q+2}{2}\right)} = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}.$$

This proves the result. □

Evaluation of integrals

Example 41. Prove that $\Gamma(n) = \int_0^1 \left(\ln \frac{1}{y}\right)^{n-1} dy,\ n > 0.$

Solution: We know that:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

Putting $x = \ln \frac{1}{y}$, i.e., $y = e^{-x}$ and $dx = -\frac{dy}{y}$, also:

$$\begin{aligned} x \to 0 &\implies y \to 1 \\ x \to \infty &\implies y \to 0. \end{aligned}$$

Hence:

$$\Gamma(n) = \int_1^0 y \left(\ln \frac{1}{y}\right)^{n-1} \cdot \left(-\frac{dy}{y}\right) = \int_0^1 \left(\ln \frac{1}{y}\right)^{n-1} dy.$$

This proves the result. □

Example 42. Prove that $\beta(n,m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{n+m}} dy = \int_0^1 \frac{x^{n-1} + x^{m-1}}{(1+x)^{n+m}} dx.$

Solution: We know that

$$\beta(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1} dx.$$

Putting $x = \frac{1}{1+y}$, i.e., $dx = -\frac{dy}{(1+y)^2}$, also:

$$\begin{aligned} x \to 0 &\implies y \to \infty \\ x \to 1 &\implies y \to 0. \end{aligned}$$

Hence:

$$\begin{aligned} \beta(n,m) &= \int_\infty^0 \left(\frac{1}{1+y}\right)^{n-1} \left(1 - \frac{1}{1+y}\right)^{m-1} \left[-\frac{dy}{(1+y)^2}\right] \\ &= \int_0^\infty \frac{y^{m-1}}{(1+y)^{n+m}} dy \\ &= \int_0^1 \frac{y^{m-1}}{(1+y)^{n+m}} dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{n+m}} dy. \end{aligned}$$

Putting $y = \frac{1}{z}$ in the second integral of the above equation we get $dy = -\frac{dz}{z^2}$, , also:

$$\begin{aligned} y \to 1 &\implies z \to 1 \\ y \to \infty &\implies z \to 0. \end{aligned}$$

Therefore:

$$\begin{aligned} \beta(n,m) &= \int_0^1 \frac{y^{m-1}}{(1+y)^{n+m}} dy + \int_1^0 \frac{\left(\frac{1}{z}\right)^{m-1}}{\left(1+\frac{1}{z}\right)^{n+m}} \left(-\frac{dz}{z^2}\right) \\ &= \int_0^1 \frac{y^{m-1}}{(1+y)^{n+m}} dy + \int_0^1 \frac{z^{n-1}}{(1+z)^{n+m}} dz \\ &= \int_0^1 \frac{x^{m-1}}{(1+x)^{n+m}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{n+m}} dx \text{ (variables are dummy)} \\ &= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{n+m}} dx. \end{aligned}$$

This proves the result. □

Example 43. Prove that $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\ln c)^{c+1}}$.

Solution: Given integral is

$$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \frac{x^c}{e^{x \ln c}} dx.$$

On putting $x \ln c = y$, i.e., $dx = \frac{dy}{\ln c}$ we get $y \to 0$ as $x \to 0$ and $y \to \infty$ as $x \to \infty$. Hence:

$$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \frac{y^c}{(\ln c)^c e^y} \cdot \frac{dy}{\ln c} = \frac{1}{(\ln c)^{c+1}} \int_0^\infty y^{c+1-1} e^{-y} dy = \frac{\Gamma(c+1)}{(\ln c)^{c+1}}.$$

This proves the result. □

Example 44. Prove that: (i) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma(1/4)}{\Gamma(3/4)}$;

(ii) $\int_0^{\pi/2} \sqrt{\tan\theta} d\theta = \frac{1}{2} \Gamma(1/4) \Gamma(3/4)$.

Solution: (i) On putting $x^2 = \sin\theta$, i.e., $x = \sqrt{\sin\theta}$ we have $dx = \dfrac{1}{2}\sin^{-1/2}\theta\cos\theta d\theta$ and

$$\begin{aligned} x \to 0 &\implies \theta \to 0 \\ x \to 1 &\implies \theta \to \pi/2. \end{aligned}$$

Hence:

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^4}} &= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} \\ &= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta d\theta \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{-1/2+1}{2}\right)}{\Gamma\left(\frac{-1/2+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \\ &= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \end{aligned}$$

(ii) Since $\displaystyle\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$, hence the given integral will be:

$$\begin{aligned} \int_0^{\pi/2} \sqrt{\tan\theta} d\theta &= \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta d\theta \\ &= \frac{\Gamma\left(\frac{1/2+1}{2}\right)\Gamma\left(\frac{-1/2+1}{2}\right)}{2\Gamma\left(\frac{1/2-1/2+2}{2}\right)} \\ &= \frac{1}{2}\Gamma(1/4)\Gamma(3/4). \end{aligned}$$

This proves the result. □

Example 45. Evaluate: (i) $\int_0^\infty a^{-bx^2} dx$; (ii) $\int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx$.

Solution: (i) Given integral is:

$$\int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-bx^2 \ln a} dx.$$

On putting $bx^2 \ln a = y$, we have $dx = \frac{dy}{2\sqrt{by \ln a}}$ and

$$\begin{aligned} x \to 0 &\implies y \to 0 \\ x \to \infty &\implies y \to \infty. \end{aligned}$$

Hence:

$$\begin{aligned} \int_0^\infty a^{-bx^2} dx &= \int_0^\infty e^{-y} \frac{dy}{2\sqrt{by \ln a}} = \frac{1}{2\sqrt{b \ln a}} \int_0^\infty e^{-y} y^{1/2-1} dy \\ &= \frac{1}{2\sqrt{b \ln a}} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{2\sqrt{b \ln a}}. \end{aligned}$$

(ii) Given integral is:

$$\int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx.$$

On putting $\ln\left(\frac{1}{x}\right) = y$, i.e., $x = e^{-y}$, we have $dx = -e^{-y} dy$ and $x \to 0 \implies y \to \infty, x \to 1 \implies y \to 0$. Hence:

$$\int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx = \int_\infty^0 e^{-4y} y^3 \left(-e^{-y} dy\right) = \int_0^\infty e^{-5y} y^3 dy.$$

Again putting $5y = z$, we have $dy = \frac{dz}{5}$ and limits of integration remains same. Hence:

$$\begin{aligned} \int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx &= \int_0^\infty e^{-z} \frac{z^3}{5^3} \cdot \frac{dz}{5} = \frac{1}{625} \int_0^\infty e^{-z} z^{4-1} dz \\ &= \frac{\Gamma(4)}{625} = \frac{3!}{625} = \frac{6}{625}. \end{aligned}$$

This is the required value. □

Example 46. Prove that $\displaystyle\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$.

Solution: Let $I_1 = \displaystyle\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}$ and $I_2 = \displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^4}}$. Then, on putting $x^2 = \sin\theta$, i.e., $x = \sin^{1/2}\theta$ in I_1, we have $dx = \dfrac{1}{2}\sin^{-1/2}\theta\cos\theta d\theta$, and $\theta \to 0$ as $x \to 0$ and $\theta \to \pi/2$ as $x \to 1$. Hence:

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2}\sin^{-1/2}\theta\cos\theta d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta d\theta \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1/2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{1/2+0+2}{2}\right)} = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \\ &= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}. \end{aligned}$$

On putting $x^2 = \tan\phi$, i.e., $x = \tan^{1/2}\phi$ in I_2, we have $dx = \dfrac{\tan^{-1/2}\phi\sec^2\phi d\phi}{2}$ and $\phi \to 0$ as $x \to 0$ and $\phi \to \pi/4$ as $x \to 1$. Hence:

$$\begin{aligned} I_2 &= \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\phi}} \cdot \frac{\tan^{-1/2}\phi\sec^2\phi d\phi}{2} = \int_0^{\pi/4} \frac{d\phi}{\sqrt{2}\sqrt{2\sin\phi\cos\phi}} \\ &= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \sin^{-1/2}(2\phi)d\phi = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\psi d\psi \quad (\text{putting } 2\phi = \psi) \\ &= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\psi\cos^0\phi d\psi \\ &= \frac{1}{2\sqrt{2}} \cdot \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-1/2+0+2}{2}\right)} \\ &= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}. \end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} &= I_1 \times I_2 \\
&= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \times \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \\
&= \frac{1}{4\sqrt{2}} \cdot \Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) \\
&= \frac{\pi}{4\sqrt{2}}.
\end{aligned}$$

This is the required result. □

Example 47. Prove that $\displaystyle\int_0^{\pi/6} \cos^6(3\theta)\sin^2(6\theta)d\theta = \frac{7\pi}{384}$.

Solution: Given integral is:

$$\begin{aligned}
I &= \int_0^{\pi/6} \cos^6(3\theta)\sin^2(6\theta)d\theta \\
&= \int_0^{\pi/6} \cos^6(3\theta)\left[2\sin(3\theta)\cos(3\theta)\right]^2 d\theta \\
&= 4\int_0^{\pi/6} \sin^2(3\theta)\cos^8(3\theta)d\theta.
\end{aligned}$$

Putting $3\theta = x$, i.e., $d\theta = \dfrac{dx}{3}$, we have $\theta \to 0 \implies x \to 0$ and $\theta \to \pi/6 \implies x \to \pi/2$. Hence:

$$\begin{aligned}
I &= 4\int_0^{\pi/2} \sin^2 x \cos^8 x \frac{dx}{3} \\
&= \frac{4}{3}\int_0^{\pi/2} \sin^2 x \cos^8 x dx \\
&= \frac{4}{3} \cdot \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{2+8+2}{2}\right)} \\
&= \frac{4}{3} \cdot \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(6)}.
\end{aligned}$$

Since, $\Gamma(n+1)=n\Gamma(n)$, $\Gamma(1)=1$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, hence we get:

$$\begin{aligned} I &= \frac{4}{3}\cdot\frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot5\cdot4\cdot3\cdot2\cdot1} \\ &= \frac{7\pi}{384}. \end{aligned}$$

This is the required result. □

Example 48. Evaluate $\int_0^{2a} x^3\sqrt{2ax-x^2}dx$.

Solution: Given integral is:

$$I = \int_0^{2a} x^3\sqrt{2ax-x^2}dx = \int_0^{2a} x^{7/2}\sqrt{2a-x}dx.$$

Putting $x=2a\sin^2\theta$, i.e., $dx=4a\sin\theta\cos\theta d\theta$, we have $x\to0 \implies \theta\to0$ and $x\to2a \implies \theta\to\pi/2$. Hence:

$$\begin{aligned} I &= \int_0^{\pi/2}(2a\sin^2\theta)^{7/2}\sqrt{2a-2a\sin^2\theta}\cdot4a\sin\theta\cos\theta d\theta \\ &= 64a^5\int_0^{\pi/2}\sin^8\theta\cos^2\theta d\theta \\ &= 64a^5\cdot\frac{\Gamma\left(\frac{8+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{8+2+2}{2}\right)} \\ &= 64a^5\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(6)}. \end{aligned}$$

Since, $\Gamma(n+1)=n\Gamma(n)$, $\Gamma(1)=1$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, hence we get:

$$\begin{aligned} I &= 64a^5\cdot\frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot5\cdot4\cdot3\cdot2\cdot1} \\ &= \frac{7\pi a^5}{8}. \end{aligned}$$

This is the required result. □

Example 49. Prove that $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx = \frac{3\pi}{128}$.

Solution: Given integral is:

$$I = \int_0^1 x^{3/2}(1-x)^{3/2}\,dx.$$

Putting $x = \sin^2\theta$, i.e., $dx = 2\sin\theta\cos\theta d\theta$, we have $\theta \to 0$ as $x \to 0$, and $\theta \to \pi/2$ as $x \to 2a$. Hence:

$$\begin{aligned} I &= \int_0^{\pi/2} (\sin^2\theta)^{3/2}\left(1-\sin^2\theta\right)^{3/2} 2\sin\theta\cos\theta d\theta \\ &= 2\int_0^{\pi/2} \sin^4\theta\cos^4\theta d\theta \\ &= 2\cdot\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+4+2}{2}\right)} \\ &= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}. \end{aligned}$$

Since, $\Gamma(n+1) = n\Gamma(n)$, $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, hence we get:

$$I = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{4\cdot 3\cdot 2\cdot 1} = \frac{3\pi}{128}.$$

This is the required result. □

Example 50. If $I_n = \int x^n(a-x)^{1/2}dx$, $n \geq 1$, then prove that:

$$(2n+3)I_n = 2anI_{n-1} - 2x^n(a-x)^{3/2}.$$

Therefore, prove that $\int_0^a x^2\sqrt{ax-x^2}dx = \frac{5\pi a^4}{28}$.

Solution: Given that:

$$I_n = \int x^n(a-x)^{1/2}dx,\ n \geq 1.$$

Using integration by parts we get:

$$\begin{aligned} I_n &= -\frac{2}{3}x^n(a-x)^{3/2} - \int nx^{n-1}\cdot\frac{2}{3}(a-x)^{3/2}(-1)dx \\ &= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}(a-x)^{3/2}dx \\ &= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2n}{3}\int x^{n-1}(a-x)^{1/2}(a-x)dx \\ &= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2na}{3}\int x^{n-1}(a-x)^{1/2}dx - \frac{2n}{3}\int x^n(a-x)^{1/2}dx \\ &= -\frac{2}{3}x^n(a-x)^{3/2} + \frac{2na}{3}I_{n-1} - \frac{2n}{3}I_n. \end{aligned}$$

Rearranging the terms we get:

$$(2n+3)I_n = 2anI_{n-1} - 2x^n(a-x)^{3/2}.$$

This is the required result.

Using the above relation for $n = 5/2, 3/2$, we obtain:

$$\begin{aligned} I_{5/2} &= \frac{1}{8}\left[5aI_{3/2} - 2x^{5/2}(a-x)^{3/2}\right] \\ &= \frac{5a}{8}I_{3/2} - \frac{1}{4}x^{5/2}(a-x)^{3/2} \\ &= \frac{5a}{8}\cdot\frac{1}{6}\left[3aI_{1/2} - 2x^{3/2}(a-x)^{3/2}\right] - \frac{1}{4}x^{5/2}(a-x)^{3/2} \\ &= \frac{5a^2}{16}I_{1/2} - \frac{5a}{24}x^{3/2}(a-x)^{3/2} - \frac{1}{4}x^{5/2}(a-x)^{3/2} \\ &= \frac{5a^2}{16}\int x^{1/2}(a-x)^{1/2}dx - \frac{5a}{24}x^{3/2}(a-x)^{3/2} - \frac{1}{4}x^{5/2}(a-x)^{3/2}. \end{aligned}$$

Now, using the above relation we obtain:

$$\begin{aligned} \int_0^a x^2\sqrt{ax-x^2}dx &= \int_0^a x^{5/2}(a-x)^{1/2}dx \\ &= \left[I_{5/2}\right]_0^a \\ &= \left[\frac{5a^2}{16}\int x^{1/2}(a-x)^{1/2}dx - \frac{5a}{24}x^{3/2}(a-x)^{3/2} - \frac{1}{4}x^{5/2}(a-x)^{3/2}\right]_0^a \\ &= \frac{5a^2}{16}\int_0^a x^{1/2}(a-x)^{1/2}dx. \end{aligned}$$

Putting $x = a\sin^2\theta$ we have $dx = 2a\sin\theta\cos\theta d\theta$, and

$$\begin{aligned} x \to 0 &\implies \theta \to 0 \\ x \to a &\implies \theta \to \pi/2. \end{aligned}$$

Hence, we have:

$$\begin{aligned}\int_0^a x^2\sqrt{ax-x^2}dx &= \frac{5a^2}{16}\int_0^{\pi/2}(a\sin^2\theta)^{1/2}(a-a\sin^2\theta)^{1/2}(2a\sin\theta\cos\theta d\theta)\\ &= \frac{5a^4}{8}\int_0^{\pi/2}\sin^2\theta(1-\sin^2\theta)^{1/2}\cos\theta d\theta\\ &= \frac{5a^4}{8}\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = \frac{5a^4}{8}\cdot\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2+2}{2}\right)}\\ &= \frac{5a^4}{16}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}.\end{aligned}$$

Since, $\Gamma(n+1)=n\Gamma(n)$, $\Gamma(1)=1$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, hence, we get:

$$\begin{aligned}\int_0^a x^2\sqrt{ax-x^2}dx &= \frac{5a^4}{16}\cdot\frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot 1}\\ &= \frac{5\pi a^4}{128}.\end{aligned}$$

This is the required result. □

Example 51. Prove that $\int_0^a (a-x)^{m-1}x^{n-1}dx = a^{m+n-1}\beta(m,n)$, where $a,n,m>0$.

Solution: Given integral is:

$$I = \int_0^a (a-x)^{m-1}x^{n-1}dx.$$

Putting $x=ay$, i.e., $dx=ady$, we have $y\to 0$ as $x\to 0$, and $y\to 1$ as $x\to a$. Hence:

$$\begin{aligned}I &= \int_0^1 (a-ay)^{m-1}(ay)^{n-1}ady\\ &= a^{m+n-1}\int_0^1(1-y)^{m-1}y^{n-1}dy\\ &= a^{m+n-1}\beta(m,n).\end{aligned}$$

This is the required result. □

Example 52. Prove that $\int_0^\infty \frac{x^8\left(1-x^6\right)}{(1+x)^{24}}dx = 0.$

Solution: Given integral is:

$$\begin{aligned} I &= \int_0^\infty \frac{x^8\left(1-x^6\right)}{(1+x)^{24}} = \int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}} \\ &= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}. \end{aligned}$$

Since, $\beta(n,m)=\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$, hence:

$$\begin{aligned} I &= \beta(9,15)-\beta(15,9) \\ &= 0. \end{aligned}$$

This is the required result. □

Example 53. Prove that $\int_0^\infty \frac{x^{n-1}}{(a+bx)^{m+n}}dx = \frac{\beta(n,m)}{a^n b^m}.$

Solution: Given integral is:

$$I = \int_0^\infty \frac{x^{n-1}}{(a+bx)^{m+n}}dx.$$

On putting $bx=ay$, i.e., $dx=\frac{a}{b}dy$, we have:

$$\begin{aligned} x\to 0 &\implies y\to 0 \\ x\to\infty &\implies y\to\infty. \end{aligned}$$

Therefore:

$$\begin{aligned} I &= \int_0^\infty \frac{a^{n-1}y^{n-1}}{b^{n-1}\left(a+ay\right)^{m+n}}\cdot\frac{a}{b}dy \\ &= \frac{1}{a^m b^n}\int_0^\infty \frac{y^{n-1}}{\left(1+y\right)^{m+n}}dy \\ &= \frac{\beta(n,m)}{a^n b^m}. \end{aligned}$$

This is the required result. □

Exercise (Assignment)

(Q.1) Show that $\int_0^\infty \frac{x^4}{4^x}dx = \frac{\Gamma(5)}{(\ln 4)^5}$.

(Q.2) It is given that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, then show that $\int_0^{\pi/2} \sqrt{\cot\theta}d\theta = \frac{\pi}{\sqrt{2}}$.

(Q.3) Show that $\displaystyle\int_0^{\pi/2} \sqrt{\sin\theta}d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}d\theta = \pi$.

(Q.4) Prove that $\displaystyle\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

(Q.5) Prove that $\displaystyle\int_0^{2a} x^3(2ax - x^2)^{3/2}dx = \frac{9\pi a^7}{16}$.

(Q.6) Prove that $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\Gamma(1/n)}{n\Gamma(1/n + 1/2)}$.

Hint: Put $x = y^{1/n}$, i.e., $dx = \frac{1}{n}y^{1/n-1}dy$, the given integral $\frac{1}{n}\int_0^1 y^{1/n-1}(1-y)^{1/2-1}dy = \frac{1}{n}\beta(1/n, 1/2)$.

(Q.7) Prove that $\displaystyle\int_0^{\pi/2} \tan^n x dx = \frac{\pi}{2\sec(n\pi/2)}$.

Hint: Use $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$, then:

$$I = \int_0^{\pi/2} \sin^n x \cos^{-n} x dx = \frac{1}{2}\beta\left(\frac{1+n}{2}, \frac{1-n}{2}\right) = \frac{1}{2}\frac{\pi}{\sin(\frac{\pi-\pi n}{2})} = \frac{\pi}{2\cos(\frac{\pi n}{2})}.$$

(Q.8) Prove that $\displaystyle\int_0^{\pi/2} x^4(1-x^2)^{3/2}dx = \frac{3\pi}{256}$.

Hint: Put $x = \sin\theta$ and then use gamma function.

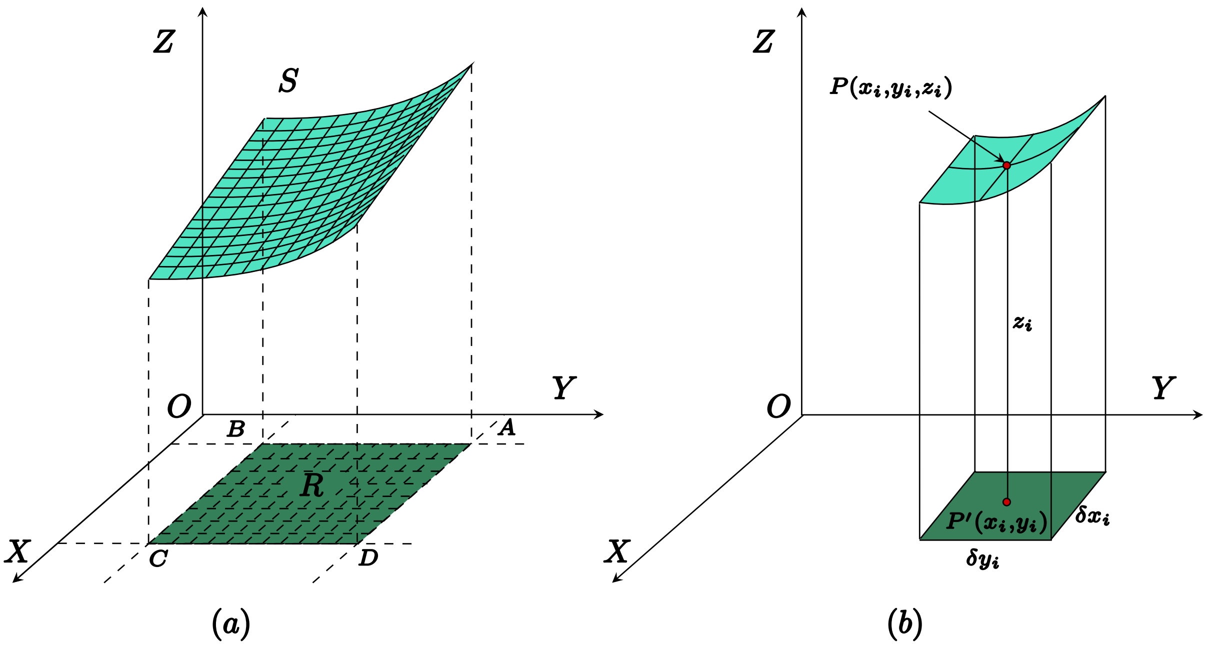
(Q.9) Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of gamma function, hence evaluate: (i) $\int_0^1 x^3(1-x^2)^4 dx$; (ii) $\int_0^1 x^5(1-x^3)^{10}dx$

Ans: $\frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$ and (i) $\frac{1}{60}$; (ii) $\frac{1}{262}$.

(Q.10) Prove that $\displaystyle\int_0^1 \frac{x^{9/2}}{\sqrt{2a-x}}dx = \frac{63\pi a^5}{8}$.

Hint: Put $x = 2a\sin^2\theta$ and then use gamma function.

Double Integral



The idea

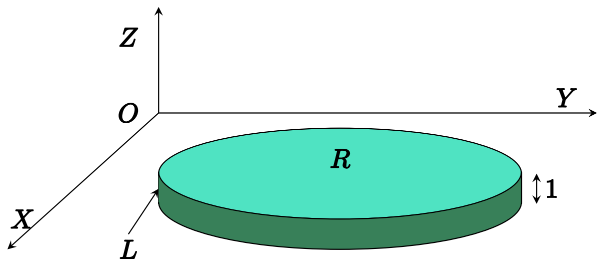
(I) Double integral as volume. Suppose, $z = f(x, y)$ represents a surface S, as shown in the figure (a). Suppose, the region R, i.e., the rectangle $ABCD$ be the region of integration. We divide this region R into a large number of small rectangles (say "n") of areas $\Delta_1 = \delta x_1 \delta y_1$, $\Delta_2 = \delta x_2 \delta y_2, \ldots, \Delta_n = \delta x_n \delta y_n$. Let $P'(x_i, y_i)$ be a point in the i^{th} area Δ_i. Let $P(x_i, y_i, z_i)$ be a point on the surface S, so that, $z_i = f(x_i, y_i)$ and $P'(x_i, y_i)$ is its projection of point P on the region R. Then, the volume of the rectangular solid of hight z_i, i.e., $f(x_i, y_i)$ and the base area $\Delta_i = \delta x_i \delta y_i$ will be $\delta v_i = z_i \delta x_i \delta y_i = f(x_i, y_i) \delta x_i \delta y_i$. Similarly, we calculate the each volume $v_1, v_2, \ldots, v_n$, and calculate the sum of volumes of all such rectangular solids thus obtained, i.e., the sum $\sum\limits_i v_i = \sum\limits_{i=1}^{n} f(x_i, y_i) \delta x_i \delta y_i$. It is clear that this sum of volumes is not exactly the volume bounded by the surface S with region R. Now, when $n \to \infty$, each small value (i.e., δ) transform into the infinitely small quantity (i.e., d). In this case, the value of the sum of volumes is called the double integral of the function f over the region R, and it is denoted by $\iint\limits_R f(x, y) dx dy$, i.e.,

$$\iint\limits_R f(x, y) dx dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \delta x_i \delta y_i.$$

It is clear that the quantity $\iint\limits_R f(x, y) dx dy$ represents the exact volume bounded by the surface S (i.e., $z = f(x, y)$) with the region R.

(II) Double integral as area. Suppose R be a given region in XY-plane and we have to find the area of this region. We consider the function $z = f(x,y) = 1$. Now it is obvious that the double integral of function $z = f(x,y) = 1$ over this region will be equal to the volume of the lamina L shown in the figure (c). Since the hight of lamina is 1, its volume, i.e., the double integral of function $z = f(x,y) = 1$ over the region R will be equal to the area of region R. Therefore:

$$\text{Area of region } R = \iint\limits_R f(x,y)dxdy = \iint\limits_R dxdy.$$



(c)

(III) Double integral as mass of lamina. Again, suppose the lamina L has a uniform surface density (mass per unit area) $\rho = \rho(x,y)$. Then the mass of the infinitely small area $dxdy$ will be $\rho(x,y)dxdy$. Therefore, the mass of the whole lamina

$$M = \iint\limits_R \rho(x,y)dxdy.$$

Similarly, for different meanings of the function $f(x,y)$ the double integral of this function over a region R can be describe in different ways. Actually, the significance of double integral is directly related to the meaning of function $f(x,y)$.

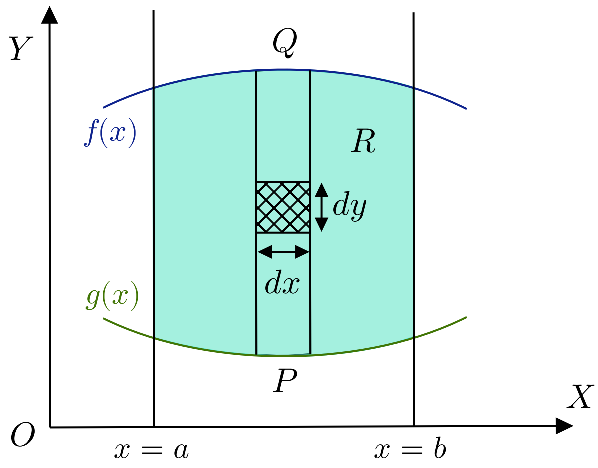
Solving double integral and limits of x and y. If the double integral is of the type $\int\limits_c^d \int\limits_a^b f(x,y)dxdy$, then we first solve the inner integral, i.e.,

$$\int\limits_c^d \int\limits_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

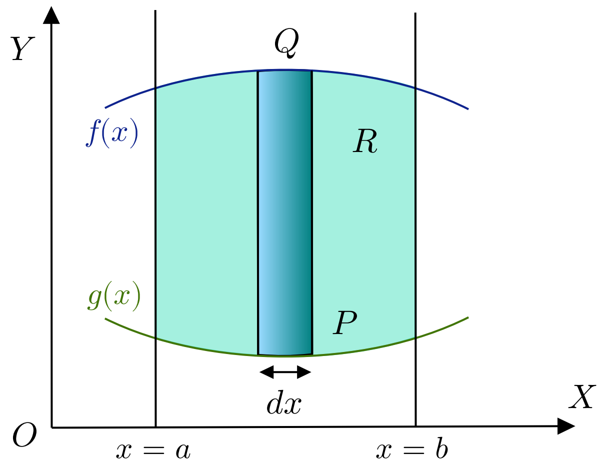
Note that, when we integrate with respect to variable x, then y must be treated as a constant and vise versa. In case, when the limit of integration is not constant, then the order of integration is decided by the variable present in the limit and we perform the first integral with respect to that variable, which is not present in the limits of inner integral. For example, in the double integral $\int\limits_a^b \int\limits_x^{x^2} f(x,y)dxdy$, we first integrate with respect to y (since y is not present in the limits of inner integral), i.e.,

$$\int_a^b \int_x^{x^2} f(x,y)dxdy = \int_a^b \left[\int_x^{x^2} f(x,y)dy \right] dx.$$

Finding the limits when the region of integration R is given.



(a)



(b)

We understand this with an example. Suppose we have to calculate the integral $\iint_R f(x,y)dxdy$, where R is the given region of integration bounded by the lines $x = a$, $x = b$ and the curves $y = f(x)$ and $y = g(x)$ as shown in figure. Since we are obtaining the limits from region R, we can integrate with respect to any variable first. Suppose, we integrate first with respect to y then x is treated as constant and we move the elementary area $dxdy$ from the bottom to the top and parallel to y axis (or along to a line parallel to y axis) in such a way that it always lies inside the region of integration. Thus, a strip PQ is formed which is parallel to y axis (if you integrate first with respect to x, then a strip parallel to y is formed). The lower end P decides the lower limit of y, and since the lower end P is situated on the curve $y = g(x)$, the lower limit of y is $y = g(x)$. Similarly, the upper end P decides the upper limit of y, and since the upper end P is situated on the curve $y = f(x)$, the upper limit of y is $y = f(x)$.

Now, we integrate with respect to x therefore y is treated as constant and now the strip PQ will move from left to right and along a line parallel to x axis in such a way that the strip always remains inside the region R and its lower end P always lie on the lower limit curve $g(x)$ and the upper end Q always lie on the upper limit curve $f(x)$. The strip moves from $x = a$ to $x = b$ to cover the whole region R, and so, the limits of x are from $x = a$ to $x = b$. Thus,

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{f(x)} f(x,y)dy \right] dx.$$

Example 54. Evaluate: $\int_0^2 \int_0^1 (x^2+y^2)dxdy$.

Solution: The given integral is:

$$\begin{aligned}\int_0^2 \int_0^1 (x^2+y^2)dxdy &= \int_0^2 \left[\int_0^1 (x^2+y^2)dx\right] dy\\ &= \int_0^2 \left[\frac{x^3}{3}+xy^2\right]_0^1 dy\\ &= \int_0^2 \left[\frac{1}{3}+y^2\right] dy\\ &= \frac{10}{3}.\end{aligned}$$

Thus, $\int_0^2 \int_0^1 (x^2+y^2)dxdy = \frac{10}{3}$. □

Example 55. Evaluate: $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dxdy$.

Solution: The given integral is:

$$\begin{aligned}\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dxdy &= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2+y^2)dy\right] dx\\ &= \int_0^1 \left[x^2y+\frac{y^3}{3}\right]_x^{\sqrt{x}} dx\\ &= \int_0^1 \left[x^{5/2}+\frac{x^{3/2}}{3}-x^3-\frac{x^3}{3}\right] dx\\ &= \frac{3}{35}.\end{aligned}$$

Thus, $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dxdy = \frac{3}{35}$. □

Example 56. Evaluate: $\int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \dfrac{dxdy}{1+x^2+y^2}$.

Solution: The given integral is:

$$\int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \int\limits_0^1 \left[\int\limits_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$$

$$= \int\limits_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \int\limits_0^1 \frac{dx}{\sqrt{1+x^2}}$$

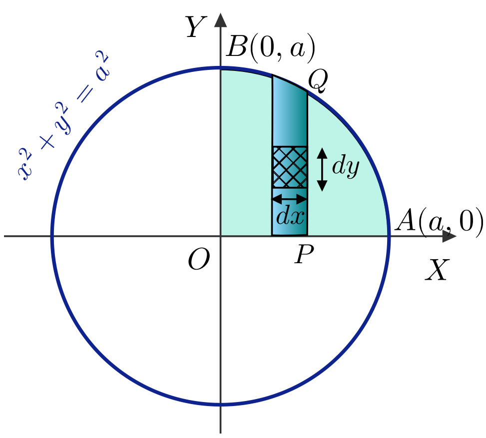
$$= \frac{\pi}{4} \left[\ln\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4} \ln(1+\sqrt{2}).$$

Thus, $\int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \dfrac{dxdy}{1+x^2+y^2} = \dfrac{\pi}{4} \ln(1+\sqrt{2})$. □

Example 57. Evaluate: $\iint\limits_R xydxdy$, where R is the positive quadrant of the circle $x^2+y^2=a^2$.

Solution: The region of integration R is the shaded part OAB in the figure. We integrate first with respect to y. Then we consider a strip PQ parallel to Y-axis lying inside the region OAB. The lower end P of strip PQ is situated on the X-axis, therefore the lower limit of y is $y=0$ (the equation of X-axis). The upper end Q is situated on the circle $x^2+y^2=a^2$, therefore the upper limit of y is $y=\sqrt{a^2-x^2}$. Now, to complete the region of integration, this strip moves from $x=0$ (i.e., the Y-axis) to the point $x=a$ (i.e., the point A), and so, the limits of x are from $x=0$ to $x=a$. Therefore:

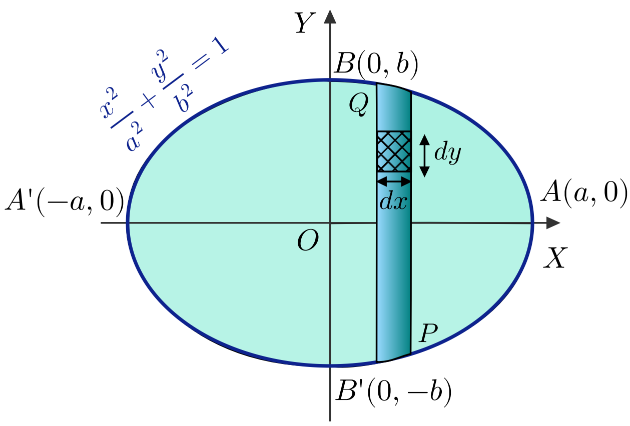


$$\iint_R xydxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xydxdy = \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} ydy\right] xdx$$
$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} xdx = \frac{1}{2}\int_0^a x(a^2-x^2)dx$$
$$= \frac{a^4}{8}.$$

Thus, $\iint_R xydxdy = \frac{a^4}{8}$. □

Example 58. Evaluate: $\iint_R (x+y)^2 dxdy$, where R is bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution: The region of integration R is the area of ellipse as shown in the figure. We integrate first with respect to y. Then we consider a strip PQ parallel to Y-axis lying inside ellipse. The lower end P of strip PQ is situated on the the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ below the X-axis, therefore the lower limit of y is $y = -\frac{b}{a}\sqrt{a^2-x^2}$. The upper end Q is again situated on the same ellipse but this time on the above of X-axis, therefore the upper limit of y is $y = \frac{b}{a}\sqrt{a^2-x^2}$. Now, to complete the region of integration, this strip moves from $x = -a$ (i.e., the point A') to the point $x = a$ (i.e., the point A), and so, the limits of x are from $x = -a$ to $x = a$. Therefore:



$$\iint_R (x+y)^2 dxdy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2 dxdy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+2xy+y^2)dxdy$$
$$= 2\int_{-a}^{a}\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)dxdy = 2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$
$$= 2\int_{-a}^{a}\left[\frac{bx^2}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$
$$= 4\int_0^{a}\left[\frac{bx^2}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx.$$

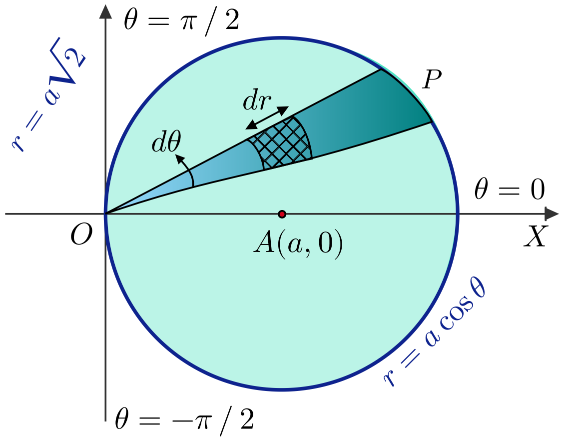
Putting $x = a\sin\theta$, the above equation reduces into the following form:

$$\begin{aligned}\iint\limits_R (x+y)^2 dxdy &= 4ab\int_0^{\pi/2}\left(a^2\sin^2\theta\cos^2\theta + \frac{b^2}{3}\cos^4\theta\right)dx\\ &= 4ab\left\{a^2\frac{\Gamma(3/2)\Gamma(3/2)}{2\Gamma(3)} + \frac{b^2}{3}\frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)}\right\}\\ &= \frac{\pi ab}{4}(a^2+b^2).\end{aligned}$$

Thus, $\iint\limits_R (x+y)^2 dxdy = \dfrac{\pi ab}{4}(a^2+b^2)$. □

Example 59. Evaluate: $\iint\limits_R r^2 d\theta dr$, where R is the are of the circle $r = a\cos\theta$.

Solution: The region of integration R is the shaded circle as shown in the figure. We integrate first with respect to r. Then we consider a strip OP along the radius vector r lying inside the region R. The lower end O of strip is situated on the pole, therefore the lower limit of r is $r = 0$. The upper end P is situated on the circle $r = a\cos\theta$, therefore, the upper limit of r is $r = a\cos\theta$. Now, to complete the region of integration, this strip rotates from $\theta = -\pi/2$ to $\theta = \pi/2$, and so, the limits of θ are from $\theta = -\pi/2$ to $\theta = \pi/2$. Therefore:

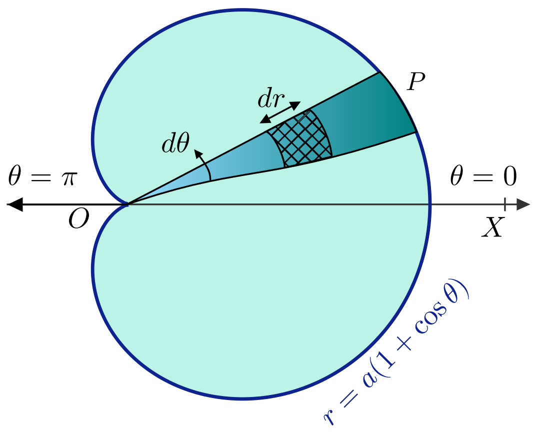


$$\begin{aligned}\iint\limits_R r^2 d\theta dr &= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r^2 d\theta dr = \int_{-\pi/2}^{\pi/2}\left[\int_0^{a\cos\theta} r^2 dr\right]d\theta\\ &= \frac{2a^3}{3}\int_0^{\pi/2}\cos^3\theta d\theta = \frac{2a^3}{3}\frac{\Gamma(2)\Gamma(1/2)}{2\Gamma(5/2)}\\ &= \frac{4a^3}{9}.\end{aligned}$$

Thus, $\iint\limits_R r^2 d\theta dr = \frac{4a^3}{9}$.

Example 60. Evaluate: $\iint\limits_R r\sin\theta d\theta dr$, where R is the region bounded by $r = a(1+\cos\theta)$ above the initial line.

Solution: The region of integration R is the shaded cardioid as shown in the figure. We integrate first with respect to r. Then we consider a strip OP along the radius vector r lying inside the region R. The lower end O of strip is situated on the pole, therefore the lower limit of r is $r = 0$. The upper end P is situated on the cardioid $r = a(1 + \cos\theta)$ therefor the upper limit of r is $r = a(1 + \cos\theta)$. Now, to complete the region of integration, this strip rotates from $\theta = 0$ to $\theta = \pi$, and so, the limits of θ are from $\theta = 0$ to $\theta = \pi$. Therefore:



$$\iint\limits_R r\sin\theta d\theta dr = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta d\theta dr = \int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} rdr\right]\sin\theta d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}\sin\theta d\theta = \frac{a^2}{2}\int_0^{\pi}\left[(1+\cos\theta)^2\sin\theta\right]d\theta.$$

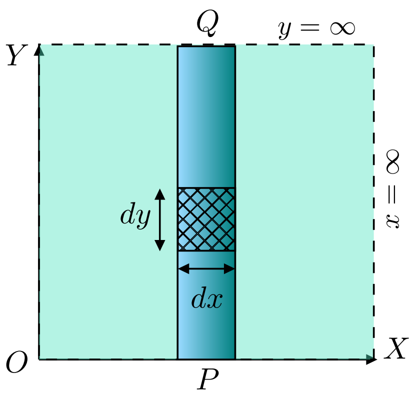
Substitute $1 + \cos\theta = t$ we obtain $\sin\theta d\theta = -dt$ and now the new limits of t are from $t = 2$ to $t = 0$. Therefore

$$\iint\limits_R r\sin\theta d\theta dr = -\frac{a^2}{2}\int_2^0 t^2 dt = -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_2^0 = \frac{4a^2}{3}.$$

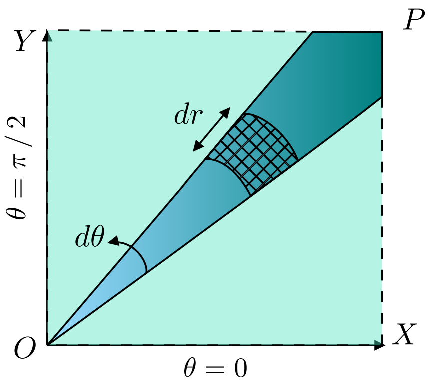
Thus, $\iint\limits_R r\sin\theta d\theta dr = \frac{4a^2}{3}$.

Example 61. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$ by transforming it into polar coordinates.

Solution:



(a) Cartesian coordinates



(b) Polar coordinates

Here, in the limits of integration, x and y both varies from 0 to ∞, therefore the region of integration is the first quadrant as shown in the figure (a). Now, to change the integral into polar coordinates, we note that the elementary area in polar coordinates is $dxdy = rdrd\theta$. Also, since $x = r\cos\theta$, $y = r\sin\theta$, we have $x^2 + y^2 = r^2$. For the new limits of r and θ we see the figure (b). We first integrate with respect to r. Then we consider a strip OP along the radius vector r and its lower end O is situated on pole and the upper end in on ∞, therefore, the limits of r are from $r = 0$ to $r = \infty$. To complete the region of integration (i.e., the first quadrant) this strip rotates from $\theta = 0$ to $\theta = \pi/2$, which are the limits of θ. Therefore:

$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy &= \int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta = \int_0^{\pi/2}\left[\int_0^\infty e^{-r^2}rdr\right]d\theta\\ &= \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta\\ &= \frac{\pi}{4}.\end{aligned}$$

Thus, $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \dfrac{\pi}{4}$. □

Exercise (Assignment)

(Q.1) Evaluate $\int\limits_0^1\int\limits_0^1 \dfrac{dydx}{\sqrt{(1-x^2)(1-y^2)}}dxdy$.

Ans: $\frac{\pi^2}{4}$.

(Q.2) Evaluate $\int\limits_0^1\int\limits_0^{x^2} e^{y/x}dxdy$.

Ans: $\frac{1}{2}$.

(Q.3) Evaluate $\int\limits_0^a\int\limits_y^a \dfrac{xdxdy}{x^2+y^2}$.

Ans: $\frac{a\pi}{4}$.

(Q.4) Evaluate $\iint\limits_R xydxdy$ over the region R, where $x + y \leq 1$ in the positive quadrant.

Ans: $\frac{1}{24}$.

(Q.5) Evaluate $\iint\limits_R xy(x+y)dxdy$ over the region R bonded by the curves $y = x^2$ and $y = x$.

Ans: $\frac{3}{56}$.

(Q.6) Prove that $\int_0^a \int_0^b \frac{dxdy}{xy} = \ln(a)\ln(b)$.

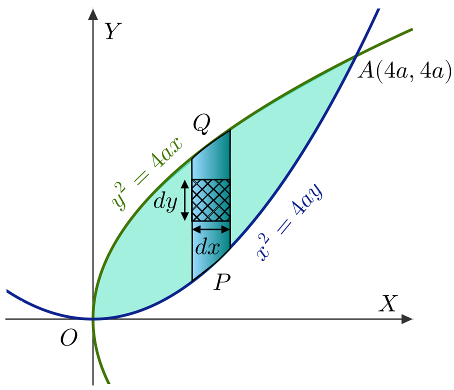
(Q.7) Prove that $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \frac{\pi}{4}\ln(1+\sqrt{2})$.

(Q.8) Prove that $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}dxdy = \frac{a^3\pi}{6}$.

Area by Double Integral

Example 62. Find the area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

Solution: The required area is the area $OPAQO$. Consider the elementary area $dxdy$ in the region $OPAQO$. We first integrate with respect to y. Then, this elementary area moves along the y axis in the region $OPAQO$ and forms the strip PQ. The lower end P of strip is situated on the parabola $x^2 = 4ay$ and the upper end Q on the parabola $y^2 = 4ax$. Therefore, the limits of y are from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$. Now, to complete the region, this strip moves along the X-axis from point O, i.e., from $x = 0$ to the point A, i.e., to $x = 4a$, therefore, the limits of x are from $x = 0$ to $x = 4a$. Thus, the required area:

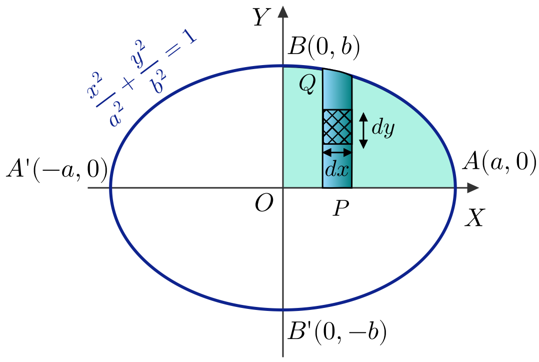


$$\text{Area } OPAQO = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dxdy = \int_0^{4a}\left[\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\right]dx = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$
$$= \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx = \frac{16a^2}{3}.$$

Hence, Area $OPAQO = \frac{16a^2}{3}$. □

Example 63. *Find the whole area the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: The required area is the shaded area and is equal to $4\times$ area $OABO$. Consider the elementary area $dxdy$ in the region $OABO$. We first integrate with respect to y. Then, this elementary area moves along the y axis in the region $OABO$ and forms the strip PQ. The lower end P of strip is situated on the X-axis, i.e., $y=0$ and the upper end Q on the arc AB of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.



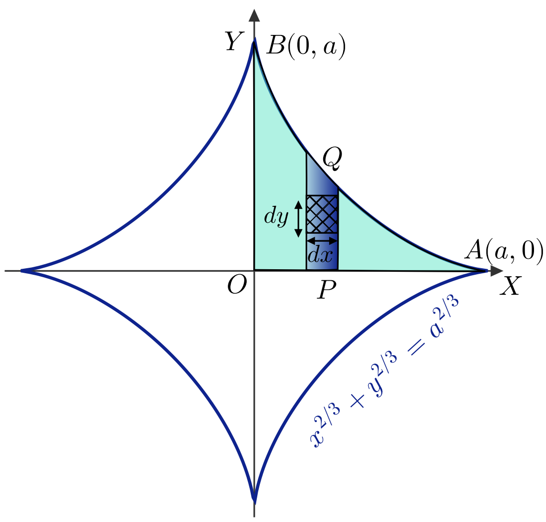
Therefore, the limits of y are from $y=0$ to $y=\frac{b}{a}\sqrt{a^2-x^2}$. Now, to complete the region, this strip moves along the X-axis from the line OB, i.e., from $x=0$ to the point A, i.e., to $x=a$, therefore, the limits of x are from $x=0$ to $x=a$. Thus, the required area:

$$\begin{aligned}
&= 4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dxdy = 4\int_0^a\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right]dx = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx\\
&= 4\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}\right)dx = 4\frac{b}{a}\int_0^a\left(\sqrt{a^2-x^2}\right)dx\\
&= \left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a\\
&= \pi ab.
\end{aligned}$$

Hence, the required area $=\pi ab$. □

Example 64. Find the whole area of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

Solution: The required area is the shaded area and is equal to $4\times$ area $OABO$. Consider the elementary area $dxdy$ in the region $OABO$. We first integrate with respect to y. Then, this elementary area moves along the y axis in the region $OABO$ and forms the strip PQ. The lower end P of strip is situated on the X-axis, i.e., $y=0$ and the upper end Q on the arc AB of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ Therefore, the limits of y are from $y=0$ to $y=\left(a^{2/3}-x^{2/3}\right)^{3/2}$. Now, to complete the region, this strip moves along the X-axis from the line OB, i.e., from $x=0$ to the point A, i.e., to $x=a$, therefore, the limits of x are from $x=0$ to $x=a$. Thus, the required area:



$$= 4\int_0^a \int_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} dxdy = 4\int_0^a \left[\int_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} dy\right] dx$$

$$= 4\int_0^a [y]_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} dx = 4\int_0^a \left(a^{2/3}-x^{2/3}\right)^{3/2} dx.$$

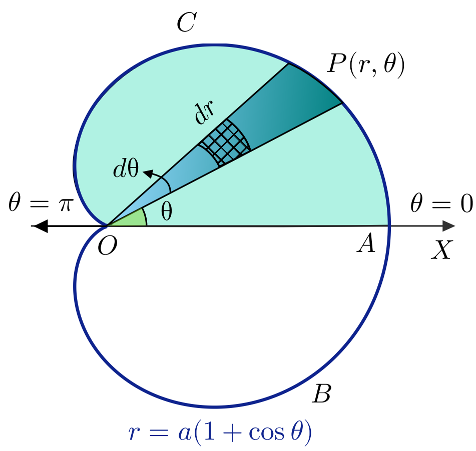
Putting $x = a\sin^3\theta$, we obtain $dx = 3a\sin^2\theta\cos\theta$ and the new limits of θ are from $\theta = 0$ to $\theta = \pi/2$. Therefore, the required area is:

$$= 12a^2 \int_0^{\pi/2} \sin^2\theta\cos^4\theta d\theta = 12a^2 \cdot \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 3!} = \frac{3\pi a^2}{8}.$$

Hence, the required area $= \dfrac{3\pi a^2}{8}$. □

Example 65. Find the whole area of the cardioid $r = a(1+\cos\theta)$.

Solution: The required area is the shaded area and is equal to $2\times$ area $OACO$. Consider the elementary area $rdrd\theta$ in the region $OACO$. We first integrate with respect to r. Then, this elementary area moves along the radial vector r in the region $OACO$ and forms the strip OP. The lower end O of strip is situated on the pole, i.e., $r = 0$ and the upper end P on the arc ACO of the cardioid $r = a(1 + a\cos\theta)$. Therefore, the limits of r are from $r = 0$ to $r = a(1 + a\cos\theta)$.



Now, to complete the region, this strip rotates from the line OX, i.e., from $\theta = 0$ to the point O, i.e., to $\theta = \pi$, therefore, the limits of θ are from $\theta = 0$ to $\theta = \pi$. Thus, the required area:

$$= 2\int_0^\pi \int_0^{a(1+\cos\theta)} rdrd\theta = 2\int_0^\pi \left[\int_0^{a(1+\cos\theta)} rdr\right] d\theta$$

$$= 2\int_0^\pi \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = a^2\int_0^\pi (1+\cos\theta)^2 d\theta$$

$$= 4a^2\int_0^\pi \cos^4\left(\frac{\theta}{2}\right) d\theta.$$

Putting $\frac{\theta}{2} = \phi$, we obtain $d\theta = 2d\phi$ and the new limits of ϕ are from $\phi = 0$ to

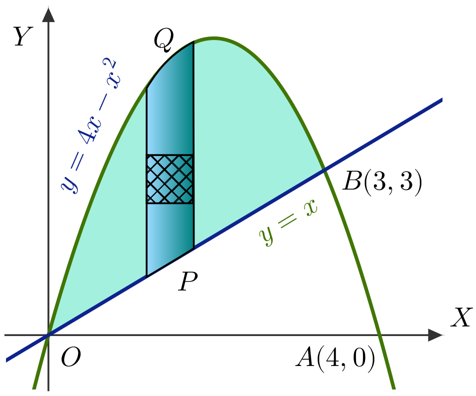
$\phi = \pi/2$. Therefore, using gamma function, the required area is:

$$\begin{aligned}
&= 8a^2 \int_0^{\pi/2} \cos^4 \phi d\phi = 8a^2 \int_0^{\pi/2} \sin^0 \phi \cos^4 \phi d\phi \\
&= \frac{\Gamma(\frac{1+0}{2})\Gamma(\frac{4+1}{2})}{2\Gamma(\frac{0+4}{2})} \\
&= 8a^2 \cdot \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{2 \cdot 2} \\
&= \frac{3\pi a^2}{2}.
\end{aligned}$$

Hence, the required area $= \dfrac{3\pi a^2}{2}$. □

Example 66. Find the area between the curves $y = 4x - x^2$ and $y = x$.

Solution: The required area is the shaded area $OPBQO$. Consider the elementary area $dxdy$ in the region $OPBQO$. We first integrate with respect to y. Then, this elementary area moves along the Y-axis in the region $OPBQO$ and forms the strip PQ. The lower end P of strip is situated on the line $y = x$ and the upper end Q on the arc OQB of the parabola $y = 4x - x^2$ Therefore, the limits of y are from $y = x$ to $y = 4x - x^2$. Now, to complete the region, this strip moves from the point O, i.e., from $x = 0$ to the point B, i.e., to $x = 3$, therefore, the limits of x are from $x = 0$ to $x = 3$. Thus, the required area:



$$\begin{aligned}
&= \int_0^3 \int_x^{4x-x^2} dxdy = \int_0^3 \left[\int_x^{4x-x^2} dy \right] dx \\
&= 2 \int_0^3 [y]_x^{4x-x^2} dx \\
&= \int_0^3 (3x - x^2)\, dx \\
&= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 \\
&= \frac{9}{2}.
\end{aligned}$$

Hence, the required area $= \frac{9}{2}$. □

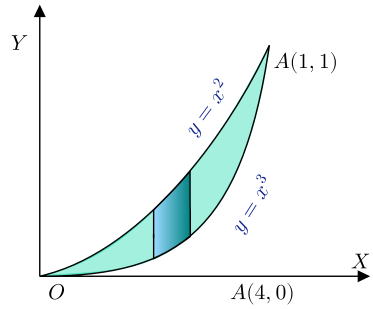
Exercise (Assignment)

(Q.1) Find the area of a circle of radius a.

(Q.2) Find the area of cardioid $r = a(1 - \cos\theta)$.

(Q.3) Find the area between the curves $y = x^2$ and $y = x^3$.

Hint: The limits of y are from $y = x^3$ to $y = x^2$ and those of x are from $x = 0$ to $x = 1$.



Triple integrals

Triple integral as mass of a solid. Suppose, the mass per unit volume (i.e., the volume density of mass) of a solid is given by its mass distribution function $w = f(x, y, z)$, where (x, y, z) represent the coordinates of points inside the solid. Let V represents the volume of the solid. Then, we divide the whole volume V into n small volumes $\delta v_i = \delta x_i \delta y_i \delta z_i$, $i = 1, 2, \ldots, n$. Suppose, $P(x_i, y_i, z_i)$ be a point in the small volume δv_i. If we choose the small volume δv_i sufficiently small, then we can assume that the volume density of mass in the volume δv_i is constant and is equal to $f(x_i, y_i, z_i)$, and so, the mass of this i^{th} small volume is given by $\delta m_i = f(x_i, y_i, z_i)\delta v_i = f(x_i, y_i, z_i)\delta x_i \delta y_i \delta z_i$. We calculate all such small masses δm_i of small volumes δv_i, where $i = 1, 2, \ldots, n$ and then sum up them and get the sum

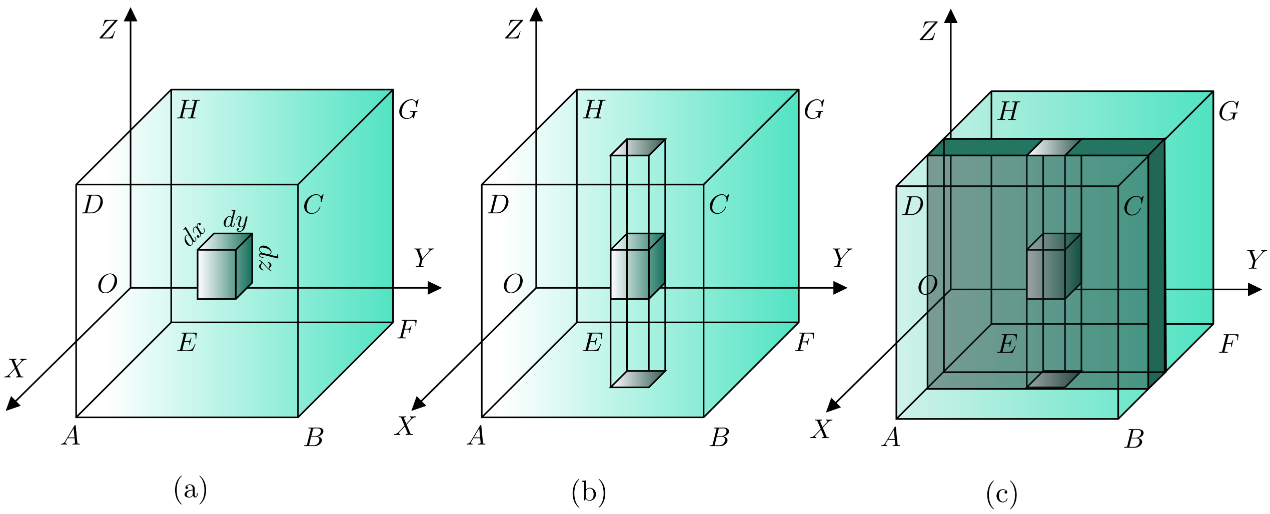
$$\sum_{i=1}^{n} \delta m_i = \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta v_i = \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta x_i \delta y_i \delta z_i.$$

It is clear that this sum masses of all small volumes is not exactly the mass of solid. Now, when $n \to \infty$, each small value (i.e., δ) transform into the infinitely small quantity (i.e., d). In this case, the value of the sum of masses is called the triple integral of the function f over the volume R, and it is denoted by $\iiint\limits_V f(x, y, z)dxdydx$, i.e.,

$$\iiint\limits_V f(x, y, z)dxdydz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\delta x_i \delta y_i \delta z_i.$$

It is clear that the quantity $\iiint\limits_V f(x, y, z)dxdydz$ represents the exact mass of the solid.

Triple integral as Volume:



Suppose, we have to find the volume of a rectangular parallelepiped (cuboid) $ABCDEFGH$ formed by the planes $x = 0, x = \alpha, y = 0, y = \beta, z = 0, z = \gamma$, i.e., $EA = \alpha, EF = \beta$ and $EH = \gamma$. Consider an infinitely small volume, i.e., the elementary volume $dv = dxdydz$ inside this cuboid (see, figure (a)). We sum up such elementary volumes along the Z-axis (i.e., during the addition the coordinates x and y remain constant), in such a way that the elementary volume remains inside the cuboid. Thus, the integration with respect to z is completed and a vertical column PQ is formed inside the cuboid $ABCDEFGH$ (see, figure (b)). The volume of this column PQ is obviously $\gamma dxdy$. Now, we sum up this column along the Y-axis (i.e., during the addition the coordinate x remains constant), in such a way that the column remains inside the cuboid. Thus, the integration with respect to y is completed and a rectangular lamina is formed inside the cuboid $ABCDEFGH$ (see, figure (c)). The volume of this rectangular lamina is obviously $\beta\gamma dx$. Finally, we sum up this rectangular lamina along the X-axis in such a way that the rectangular lamina remains inside the cuboid. Thus, the integration with respect to x is completed and the whole cuboid $ABCDEFGH$ is formed and we get the volume of cuboid, i.e., $V = \alpha\beta\gamma$. Thus:

$$\begin{aligned} V &= \iiint\limits_V dv \\ &= \int_{x=0}^{\alpha}\int_{y=0}^{\beta}\int_{z=0}^{\gamma} dxdydz. \end{aligned}$$

Example 67. Evaluate: $\int_2^4 \int_{y=0}^{x} \int_{z=0}^{x+y} z dxdydz$.

Solution: Given integral can be solved as follows:

$$\begin{aligned}
\int_2^4 \int_{y=0}^{x} \int_{z=0}^{x+y} z dx dy dz &= \int_2^4 \int_{y=0}^{x} \left[\int_{z=0}^{x+y} z dz\right] dx dy \\
&= \int_2^4 \int_{y=0}^{x} \left[\frac{z^2}{2}\right]_0^{x+y} dx dy \\
&= \int_2^4 \left[\int_0^x \frac{(x+y)^2}{2} dy\right] dx \\
&= \int_2^4 \left[\frac{(x+y)^3}{6}\right]_0^x dx \\
&= \int_2^4 \left[\frac{8x^3}{6} - \frac{x^3}{6}\right] dx \\
&= 70.
\end{aligned}$$

Thus, $\int_2^4 \int_{y=0}^{x} \int_{z=0}^{x+y} z dx dy dz = 70.$ □

Example 68. Evaluate: $\int_1^e \int_1^{\ln(y)} \int_1^{e^x} \ln(z) dx dy dz.$

Solution: Given integral can be solved as follows:

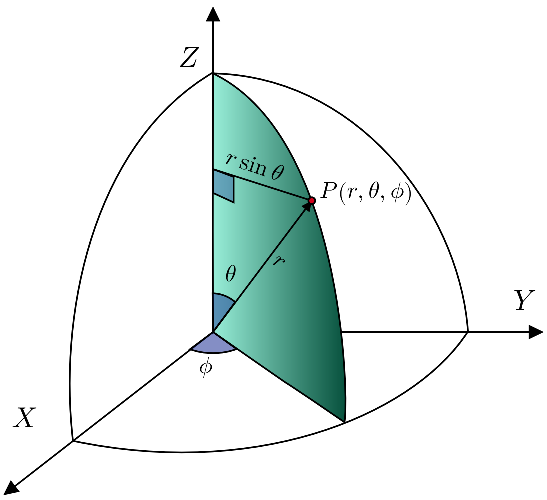
$$\begin{aligned}
I &= \int_1^e \int_1^{\ln y} \int_1^{e^x} \ln z dx dy dz \\
&= \int_1^e \int_1^{\ln y} \left[\int_1^{e^x} \ln z dz\right] dx dy \\
&= \int_1^e \int_1^{\ln y} [z \ln z - z]_1^{e^x} dx dy \\
&= \int_1^e \int_1^{\ln y} [(x-1)e^x + 1] dx dy \\
&= \int_1^e \left[\int_1^{\ln y} (x-1)e^x + 1 dx\right] dy \\
&= \int_1^e [(x-1)e^x - e^x + x]_1^{\ln y} dy \\
&= \int_1^e [(y+1)\ln y - 2y + e - 1] dy \\
&= \left[\left(\frac{y^2}{2} + y\right)\ln y - \frac{1}{y}\left(\frac{y^2}{2} + y\right) - y^2 + (e-1)y\right]_1^e \\
&= \frac{1}{4}(e^2 - 8e + 13).
\end{aligned}$$

Thus, $\int_1^e \int_1^{\ln(y)} \int_1^{e^x} \ln(z)dxdydz = \frac{1}{4}(e^2 - 8e + 13).$ $\square$

Spherical coordinates. The spherical coordinates of a point P in the space are given by $P(r, \theta, \phi)$ (see the figure below). The relation between cartesian and spherical coordinates is given by:

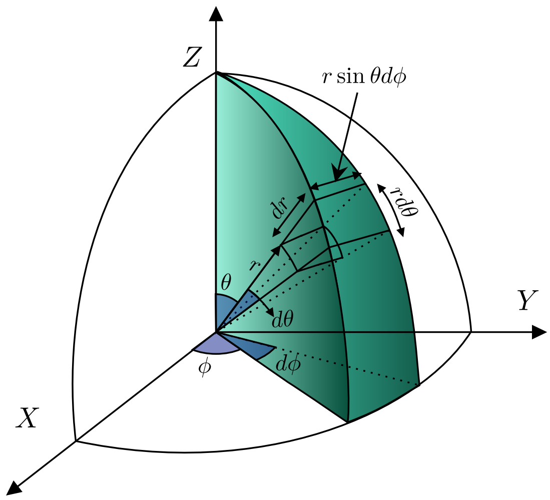
$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta.$$

The elementary volume in spherical coordinates is given by $dV = r^2\sin\theta drd\theta d\phi$ (see the figure below). Some times it is easy to solve the integral by changing the cartesian coordinates into spherical coordinates. There is no hard and fast rule to decide whether it is easy to use the spherical coordinates, we can decide only by observations. Although, converting to spherical coordinates can make triple integrals much easier to work out when the region you are integrating over has some spherical symmetry.



Spherical coordinates

fx



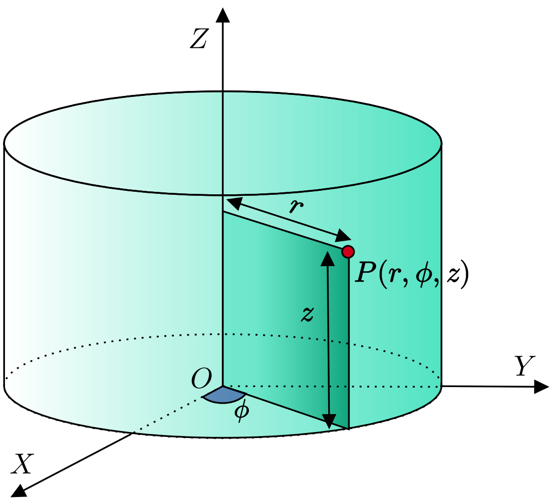
Elementary volume in spherical coordinates

$dV = r^2\sin\theta dr\theta d\phi$

Cylindrical coordinates. The cylindrical coordinates of a point P in the space are given by $P(r, \phi, z)$ (see the figure below). The relation between cartesian and spherical coordinates is given by:

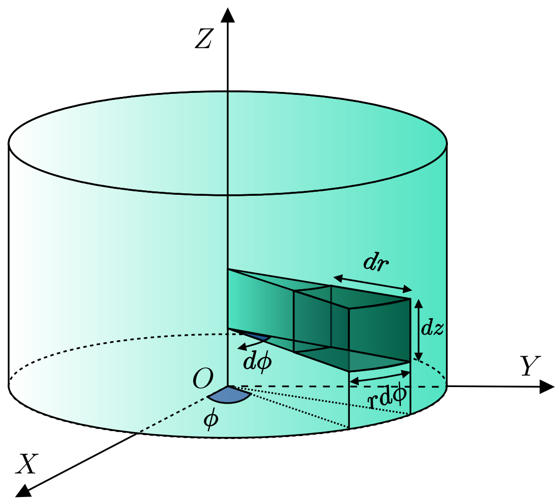
$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z.$$

The elementary volume in cylindrical coordinates is given by $dV = rdrd\phi dz$ (see the figure below). Some times it is easy to solve the integral by changing the cartesian coordinates into cylindrical coordinates. There is no hard and fast rule to decide whether it is easy to use the cylindrical coordinates, we can decide only by observations. Although, converting to cylindrical coordinates can make triple integrals much easier to work out when the region you are integrating over has some cylindrical symmetry.



Cylindrical coordinates

$OP = r,\ QP = z,\ \angle QOX = \phi$



Elementary volume in cylindrical coordinates

$dV = rdrd\phi dz$

Example 69. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: The volume of the given sphere is: $V = 8 \iiint_{V_p} dxdydz$ where V_p is the volume of the sphere in the positive octant. Changing the coordinates into the spherical coordinates we get:

$$dxdydz = r^2 \sin\theta drd\theta d\phi$$

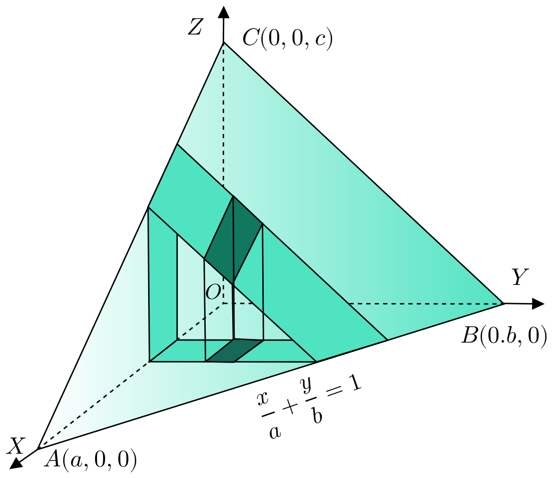
and in the volume V_p, r changes from $r = 0$ to a, θ changes from $\theta = 0$ to $\pi/2$ and ϕ changes from $\phi = 0$ to $\pi/2$. Thus, the required volume:

$$\begin{aligned} V &= 8 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{a} r^2 \sin\theta drd\theta d\phi = \frac{8a^3}{3} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta d\theta d\phi \\ &= \frac{8a^3}{3} \int_{\phi=0}^{\pi/2} d\phi = \frac{4\pi a^3}{3}. \end{aligned}$$

Thus, the volume of a sphere of radius a is $\dfrac{4\pi a^3}{3}$. □

Example 70. Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Solution: The required volume is the volume of the tetrahedron as shown in the figure. The value of z changes from $z = 0$ to $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ so that a bar from $z = 0$ to $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ is created. Then y changes from $y = 0$ to $y = b\left(1 - \frac{x}{a}\right)$ and a triangular plane is thus created. Finally, x changes from $x = 0$ to $x = a$ and the volume is completed. Thus, the required volume:

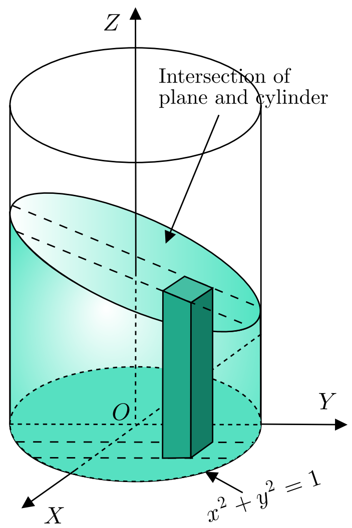


$$\begin{aligned} V &= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dxdydz = \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} [z]_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dxdy \\ &= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dxdy = \int_{x=0}^{a} c\left[\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)} dx \\ &= \frac{bc}{2}\int_{x=0}^{a}\left(1-\frac{x}{a}\right)^2 dx = \frac{abc}{6}. \end{aligned}$$

□

Example 71. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution: Suppose the required volume is V which is the dark shaded part in the figure. Then $V = \iiint\limits_V dxdydz$. The limits of z are from $z = 0$ to $z = 4 - y$ and then x and y varies according to the limits of circle $C : x^2 + y^2 = 4$. The variable y varies from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$, and then x varies from $x = -2$ to $x = 2$. Thus, the required volume:



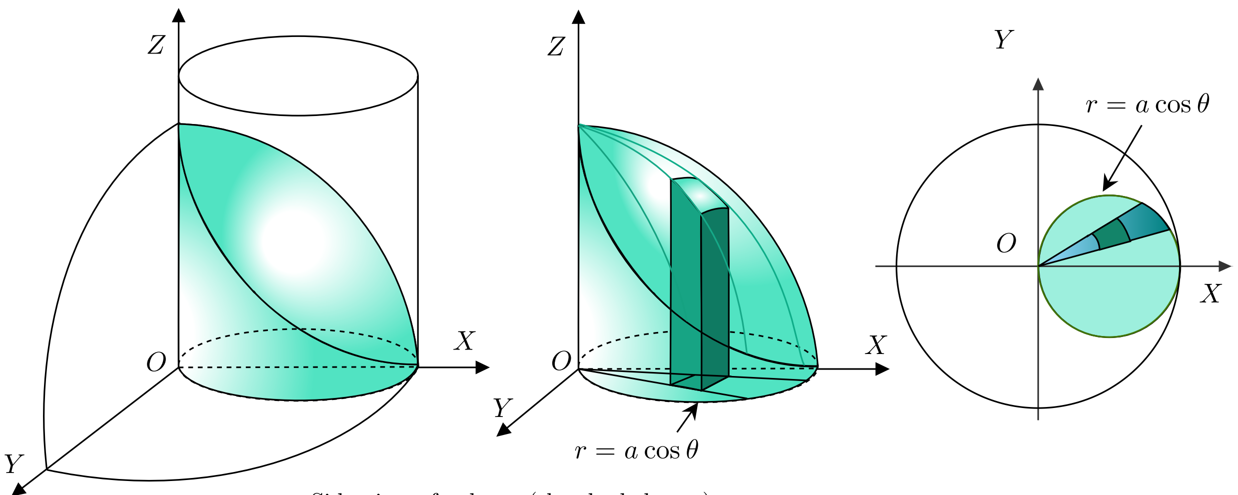
$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{4-y} dxdydz$$

$$\begin{aligned} &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)dxdy = \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)dy\right] dx = 2\int_{-2}^{2}\left[\int_{0}^{\sqrt{4-x^2}} 4dy\right] dx \\ &= 8\int_{-2}^{2}\left[\sqrt{4-x^2}\right] dx = 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 dx = 16\pi. \end{aligned}$$

Thus, the required volume $V = 16\pi$. □

Example 72. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

Solution:



Side view of volume (the shaded part) Top view

The required volume V is the 2 times of the volume V_1 shown in the figure by shaded part (half of the volume is above of the xy-plane and half is below). To make easy, we will use cylindrical coordinates. The cylindrical coordinates are:

$$\begin{aligned} x &= r\cos\theta; \\ y &= r\sin\theta; \\ z &= z. \end{aligned}$$

and the volume element $dV = rdrd\theta dz$. The equation of sphere will become $r^2 + z^2 = a^2$, and the equation of cylinder will become $r = a\cos\theta$. In the shaded part, the limits of z are from $z = 0$ to $z = \sqrt{a^2 - r^2}$ and the r and θ varies throughout the circle $r = a\cos\theta$ (see the figure). Therefore, r varies from $r = 0$ to $r = a\cos\theta$ and θ from $\theta = -\pi/2$ to $\theta = \pi/2$. Thus, the limits of r, θ and z are:

$$\begin{aligned} z &= 0 \text{ to } z = \sqrt{a^2 - r^2}; \\ r &= 0 \text{ to } r = a\cos\theta; \\ \theta &= \frac{\pi}{2} \text{ to } \theta = \frac{\pi}{2}. \end{aligned}$$

Therefore, the required volume:

$$V = 2\iiint_{V_1} dxdydz = \iiint_{V_1} rdrd\theta dz.$$

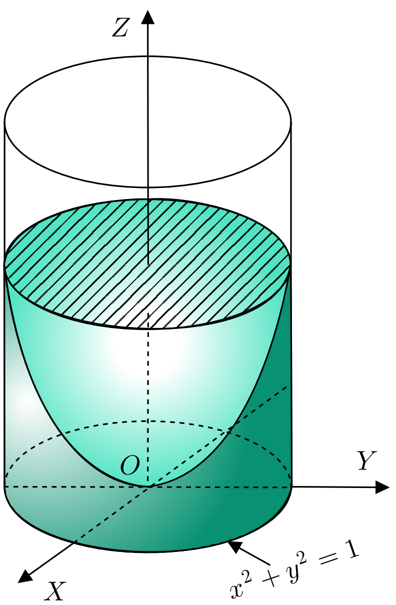
Applying the limits for r, θ and z we get:

$$\begin{aligned}
V &= 2\int_{-\pi/2}^{\pi/2}\int_{0}^{a\cos\theta}\int_{0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz = 2\int_{-\pi/2}^{\pi/2}\int_{0}^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta \\
&= 2\int_{-\pi/2}^{\pi/2}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^{a\cos\theta} d\theta = \frac{2a^3}{3}\int_{-\pi/2}^{\pi/2}\left(1-\sin^3\theta\right)d\theta \\
&= \frac{4a^3}{3}\int_{0}^{\pi/2}\left(1-\sin^3\theta\right)d\theta = \frac{4a^3}{3}\int_{0}^{\pi/2}\left[1-\sin\theta\left(1-\cos^2\theta\right)\right]d\theta \\
&= \frac{4a^3}{3}\int_{0}^{\pi/2}\left[1-\sin\theta+\sin\theta\cos^2\theta\right]d\theta = \frac{4a^3}{3}\left[\theta+\cos\theta-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} \\
&= \frac{4a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).
\end{aligned}$$

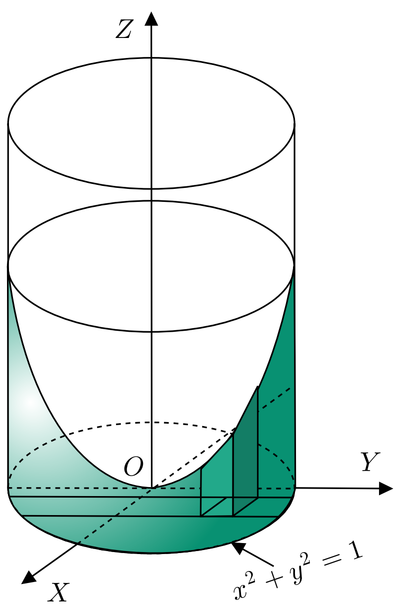
Thus, the required volume $V = \frac{4a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$. □

Example 73. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.

Solution:



The dark shaded region is the required volume



The required volume

Suppose the required volume is V. Then $V = 4\iiint_V dxdydz$. The limits of z are from $z = 0$ to $z = \dfrac{x^2+y^2}{2}$ and then x and y varies according to the limits of circle $C : x^2 + y^2 = 4$. The variable y varies from $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$, and then x varies from $x = -2$ to $x = 2$. Thus, the required volume:

$$\begin{aligned} V &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{\frac{x^2+y^2}{2}} dxdydz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{x^2+y^2}{2} dxdy = \int_{-2}^{2}\left[\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)dy\right] dx \\ &= \int_{-2}^{2}\left[x^2y + \frac{y^3}{3}\right]_0^{\sqrt{4-x^2}} dx = 2\int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}\left(4-x^2\right)^{3/2}\right] dx. \end{aligned}$$

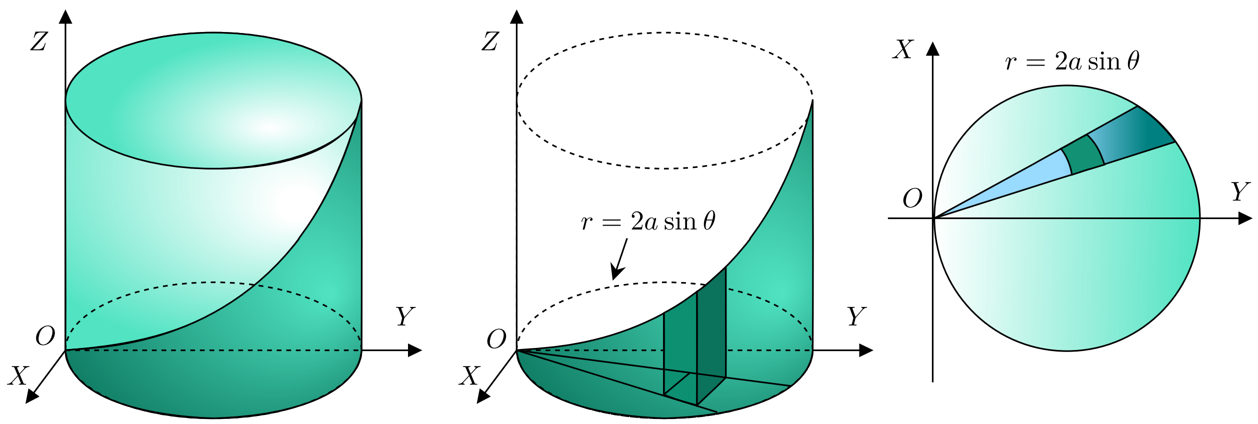
Putting $x = 2\sin\theta$, the limits now changed to $\theta = 0$ to $\theta = \pi/2$, and $dx = 2\cos\theta d\theta$. Therefore:

$$\begin{aligned} V &= 16\int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta + \frac{16}{3}\int_0^{\pi/2} \sin^0\theta\cos^4\theta d\theta \\ &= 16\cdot\frac{\Gamma(3/2)\Gamma(3/2)}{2\cdot\Gamma(3)} + \frac{16}{3}\cdot\frac{\Gamma(1/2)\Gamma(5/2)}{2\cdot\Gamma(3)} \\ &= 4\pi. \end{aligned}$$

Thus, the required volume is $V = 4\pi$. □

Example 74. Find the volume between the paraboloid $x^2 + y^2 = az$, cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

Solution:



Side view of volume (the dark shaded part)   Top view

The required volume V is shown in the figure by shaded part. To make easy, we will use cylindrical coordinates. The cylindrical coordinates are: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and the volume element $dV = rdrd\theta dz$. The equation of paraboloid will become $r^2 = az$, and the equation of cylinder will become $r = 2a\sin\theta$. In the shaded part, the limits of z are from $z = 0$ to $z = \frac{r^2}{a}$ and the r and θ varies throughout the circle $r = 2a\sin\theta$ (see the figure). Therefore, r varies from $r = 0$ to $r = 2a\sin\theta$ and θ from $\theta = 0$ to $\theta = \pi$. Therefore, the required volume:

$$\begin{aligned} V &= \iiint_V dxdydz = \iiint_V rdrd\theta dz = \int_0^{\pi}\int_0^{2a\sin\theta}\int_0^{r^2/a} rdrd\theta dz \\ &= \int_0^{\pi}\int_0^{2a\sin\theta} \frac{r^3}{a} drd\theta = \frac{1}{4a}\int_0^{\pi} \left[r^4\right]_0^{2a\sin\theta} d\theta \\ &= 4a^3\int_0^{\pi} \left(\sin^4\theta\right) d\theta = 8a^3\int_0^{\pi/2} \left(\sin^4\theta\cos^0\theta\right) d\theta \\ &= 8a^3 \cdot \frac{\Gamma(5/2)\Gamma(1/2)}{2\cdot\Gamma(3)} = 8a^3 \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 2} \\ &= \frac{3\pi a^3}{3}. \end{aligned}$$

Thus, the required volume $V = \dfrac{3\pi a^3}{3}$. □

Exercise (Assignment)

(Q.1) Evaluate: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x + y + z)dxdydz$.

Ans. 0

(Q.2) Evaluate: $\int_{0}^{\ln 2}\int_{0}^{x}\int_{0}^{x+\ln y} e^{x+y+z}dxdydz$.

Ans. $\dfrac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$.

(Q.3) Evaluate: $\iiint_R (x - 2y + z)dxdydz$, where R is the region determine by $0 \le x \le 1$, $0 \le y \le x^2, 0 \le z \le x + y$.

Ans. $\dfrac{29}{105}$.

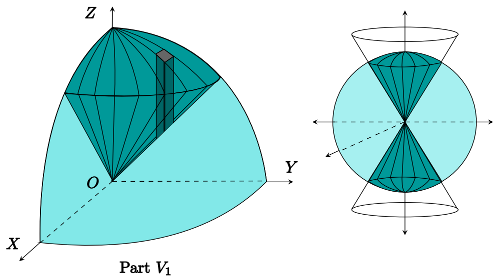
(Q.4) Evaluate: $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \dfrac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$. (Hint: From the limits of integration, it is clear that the region of integration is the part of sphere

$x^2+y^2+z^2=1$ in the positive quadrant. Change the cartesian coordinates into the spherical coordinates, the new limits will be $r=0$ to 1, $\phi=0$ to $\pi/2$ and $\theta=0$ to $\pi/2$.)

Ans. $\frac{\pi^2}{8}$.

(Q.5) Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$. (A filled Ice-cream cone)

Hint. Suppose the required volume is V. Then $V=8\iiint_{V_1} dxdydz$ where V_1 is part of volume in the positive octant and it is the shaded part in the figure. The spherical coordinates are $x=r\cos\phi\sin\theta, y=r\sin\phi\sin\theta, z=r\cos\theta$. Equation of sphere $r^2=a^2$ and of cone is $\tan^2\theta=1$, i.e., $\theta=\pm\frac{\pi}{4}$. The limits of r are from $r=0$ to $r=a$ and then ϕ varies from $\phi=0$ to 2π and θ varies from $\theta=0$ to $\frac{\pi}{4}$. Thus, the required volume is $\frac{2\pi a^2}{3}(2-\sqrt{2})$.



Part V_1

♣ ♣ ♣ ♣

Unit-V

Linear differential equations of n$^{\text{th}}$ order: Linear differential equations of n$^{\text{th}}$ order, method of variation of parameter and Cauchys homogenous linear equations.

Linear differential equations of n$^{\text{th}}$ order

The following differential equation:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x). \tag{4}$$

is called an ordinary linear differential equations of nth order.

There are two parts of solution of differential equation (4). One is called the Complementary function or C.F., which is the solution of the following homogeneous equation:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0. \tag{5}$$

The second part is the particular solution or P.I., which is a solution of differential equation (4) with no arbitrary constant, and the complete solution of differential equation (4) is given by:

$$y = \text{C.F.+P.I.}$$

Definition 4. *The solution of homogeneous equation* (5) *which consist* n *arbitrary constant is called the complementary function (C.F.) of equation* (4).

Rules for finding C.F.: There are n linearly independent solutions of differential equation (5), say, $y_1, y_2, \ldots, y_n$ and the C.F. will be the linear combination of all these solutions, i.e.,

$$C.F. = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

To find the solutions $y_1, y_2, \ldots, y_n$, we denote $\frac{d}{dx}$ by D, i.e., $D \equiv \frac{d}{dx}$, differential equation (4) will be

$$\begin{aligned} (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y &= \phi(x) \\ \implies F(D) y &= \phi(x) \end{aligned} \tag{6}$$

where $F(D) \equiv a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$. Now find the roots of equation:

$$F(m) = 0 \text{ (auxiliary equation)}.$$

Suppose, $m_1, m_2, \ldots, m_n$ are the roots of auxiliary equation. Then:

(A) If all the roots of auxiliary equation are real and distinct, then

$$\text{C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

(B) If roots are real but k roots are equal, i.e., $m_1 = m_2 = \cdots = m_k$ and remaining roots are distinct, then

$$\text{C.F.} = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{m_1 x} + c_{k+1}e^{m_{k+1}x} + \cdots + c_n e^{m_n}.$$

(C) If there is any pair of complex roots, say, $\alpha \pm i\beta$, then the corresponding part of C.F. will be

$$= e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)].$$

(D) If we get two pair of complex roots equal, i.e., $\alpha_1 \pm i\beta_1 = \alpha_2 \pm i\beta_2 = \alpha \pm i\beta$ (say), then the corresponding C.F. will be

$$= e^{\alpha x}[(c_1 + c_2 x)\cos(\beta x) + (c_3 + c_4 x)\sin(\beta x)].$$

If k pair of complex roots are equal, then the above formula can be generalized in similar way.

Next, we consider techniques for finding the P.I.

Definition 5. *The function* $\dfrac{1}{F(D)}\,\phi(x)$ *satisfies the equation* (6), *therefore it is a solution of equation* (4), *and since it is free of arbitrary constant, it is called a particular solution.*

Example 75. Prove the following results:

(I) $\dfrac{1}{D}\phi(x) = \int \phi(x)dx.$

(II) $\dfrac{1}{D-m}\phi(x) = e^{mx}\int e^{-mx}\phi(x)dx.$

Solution: (I) Let

$$\frac{1}{D}\phi(x) = y$$

then we have $\phi(x) = Dy$, i.e., $\dfrac{dy}{dx} = \phi(x)$. On integrating with respect to x, we obtain

$$\begin{aligned} y &= \int \phi(x)dx \\ \Longrightarrow \frac{1}{D}\phi(x) &= \int \phi(x)dx. \end{aligned}$$

(II) Let

$$\frac{1}{D-m}\phi(x) = y$$

then we have $\phi(x) = (D-m)y$, i.e., $\dfrac{dy}{dx} - my = \phi(x)$. It is a linear equation in y.

Here, I.F.$= e^{\int -m dx} = e^{-mx}$ and the solution will be:

$$ye^{-mx} = \int e^{-mx}\phi(x)dx \text{ (without constant } c \text{ because P.I. consists no constant)}$$
$$\implies \quad y = e^{mx}\int e^{-mx}\phi(x)dx$$
$$\implies \quad \frac{1}{D-m}\phi(x) = e^{mx}\int e^{-mx}\phi(x)dx.$$

This is the required value. □

Short-cut Methods for finding P.I.: The P.I. of differential equation (4) is given by:

$$\text{P.I.} = \frac{1}{F(D)}\phi(x).$$

(a) If $\phi(x) = e^{ax}$, then

$$\text{P.I.} = \frac{1}{F(D)}\phi(x) = \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$$

provided $F(a) \neq 0$.

Note. If in the case (a) we have $F(a) = 0$, it implies that $D - a$ is a factor of $F(D)$, then we write

$$\begin{aligned}\text{P.I.} &= \frac{1}{F(D)}\phi(x) = \frac{1}{(D-a)G(D)}e^{ax} = \frac{1}{(D-a)G(a)}e^{ax}\\ &= \frac{1}{G(a)} \cdot \frac{1}{D-a}e^{ax}.\end{aligned}$$

provided $G(a) \neq 0$. If $G(a) = 0$, we again repeat this process.

(b) If $\phi(x) = e^{ax} \cdot f(x)$, then

$$\text{P.I.} = \frac{1}{F(D)}\phi(x) = \frac{1}{F(D)}e^{ax} \cdot f(x) = e^{ax}\frac{1}{F(D+a)}f(x).$$

(c) If $\phi(x) = \sin(ax)$ (or $\phi(x) = \cos(ax)$) then

$$\text{P.I.} = \frac{1}{F(D^2)}\phi(x) = \frac{1}{F(D^2)}\sin(ax) = \frac{1}{F(-a^2)}\sin(ax)$$

provided $F(-a^2) \neq 0$. The formula for $\cos(ax)$ is same.

Note. If $F(D) \equiv D^2 + a^2$, then $F(-a^2) = 0$. In this case we can use the following direct formula:

$$\frac{1}{D^2+a^2}\sin(ax) = -\frac{x}{2a}\cos(ax), \quad \frac{1}{D^2+a^2}\cos(ax) = \frac{x}{2a}\sin(ax).$$

(d) If $\phi(x) = x^n$, then we write $F(D)$ in form of $1 \pm G(D)$ and then we use one of the following expressions:

$$\begin{aligned}(1-a)^{-1} &= 1 + a + a^2 + a^3 + \cdots \\ (1+a)^{-1} &= 1 - a + a^2 - a^3 + \cdots, \quad \text{then}\end{aligned}$$

$$\text{P.I.} = \frac{1}{F(D)}\phi(x) = \frac{1}{1 \pm G(D)}x^n = [1 \pm G(D)]^{-1}\, x^n.$$

Now, use the expansion of $[1 \pm G(D)]^{-1}$.

(e) If $\phi(x) = x \cdot f(x)$, then

$$\text{P.I.} = \frac{1}{F(D)}x \cdot f(x) = x \cdot \left[\frac{1}{F(D)} \cdot f(x)\right] + \frac{d}{dD}\left[\frac{1}{F(D)}\right] \cdot f(x).$$

Example 76. Solve: $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0.$

Solution: Putting $D \equiv \dfrac{d}{dt}$, given differential equation can be written as:

$$(D^2 + 5D + 6)x = 0 \implies F(D)x = 0$$

where $F(D) \equiv D^2 + 5D + 6$. The auxiliary equation of the above equation will be:

$$\begin{aligned}F(m) = 0 &\implies m^2 + 5m + 6 = 0 \\ &\implies (m+2)(m+3) = 0 \\ &\implies m = -2, -3.\end{aligned}$$

Therefore, the roots of the auxiliary equation are real and distinct, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}.$$

Since the given equation is homogeneous, therefore, C.F. is the solution of given equation, i.e., $y = c_1 e^{-2x} + c_2 e^{-3x}$. □

Example 77. Solve: $(D^4 - 4D + 4)y = 0.$

Solution: Given differential equation can be written as:

$$F(D)x = 0$$

where $F(D) \equiv D^4 - 4D + 4$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^4 - 4m^2 + 4 = 0 \\ &\implies \quad (m^2 - 2)^2 = 0 \\ &\implies \quad m^2 = 2, 2 \\ &\implies \quad m = \pm\sqrt{2}, \pm\sqrt{2} \\ &\implies \quad m = \sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}. \end{aligned}$$

Therefore, the roots of the auxiliary equation are real and in two pairs of equal roots, and so, the complementary function will be:

$$\text{C.F.} = (c_1 + c_2 x)e^{\sqrt{2}x} + (c_3 + c_4 x)e^{-\sqrt{2}x}.$$

Since the given equation is homogeneous, therefore, C.F. is the solution of given equation, i.e., $y = (c_1 + c_2 x)e^{\sqrt{2}x} + (c_3 + c_4 x)e^{-\sqrt{2}x}$. □

Example 78. Solve: $\dfrac{d^4y}{dx^4} + m^4 y = 0.$

Solution: Given differential equation can be written as:

$$(D^4 + m^4)y = 0 \implies F(D)x = 0$$

where $F(D) \equiv D^4 + m^4$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(M) = 0 \quad &\implies \quad M^4 + m^4 = 0 \\ &\implies \quad (M^2 + m^2)^2 = 2M^2m^2 \\ &\implies \quad M^2 + m^2 = \pm\sqrt{2}Mm \\ &\implies \quad M^2 + m^2 - \sqrt{2}Mm = 0,\ M^2 + m^2 + \sqrt{2}Mm = 0 \\ &\implies \quad M = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}},\ M = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}. \end{aligned}$$

Therefore, the roots of the auxiliary equation are complex, and so, the complementary function will be:

$$\text{C.F.} = e^{mx/\sqrt{2}}\left(c_1 \cos\frac{mx}{\sqrt{2}} + c_2 \sin\frac{mx}{\sqrt{2}}\right) + e^{-mx/\sqrt{2}}\left(c_3 \cos\frac{mx}{\sqrt{2}} + c_4 \sin\frac{mx}{\sqrt{2}}\right).$$

Since the given equation is homogeneous, therefore, C.F. is the solution of given equation, i.e.:

$$y = e^{mx/\sqrt{2}}\left(c_1 \cos\frac{mx}{\sqrt{2}} + c_2 \sin\frac{mx}{\sqrt{2}}\right) + e^{-mx/\sqrt{2}}\left(c_3 \cos\frac{mx}{\sqrt{2}} + c_4 \sin\frac{mx}{\sqrt{2}}\right). \quad \square$$

Example 79. Solve: $(D^2 + 5D + 6)y = e^x$.

Solution: Given differential equation can be written as:

$$F(D)y = \psi(x)$$

where $F(D) \equiv D^2 + 5D + 6$ and $\psi(x) = e^x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 + 5m + 6 = 0 \\ &\implies \quad (m+2)(m+3) = 0 \\ &\implies \quad m = -2, -3 \end{aligned}$$

Therefore, the roots of the auxiliary equation are real and distinct, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}.$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2 + 5D + 6}\,e^x \\ &= \frac{1}{1^2 + 5\cdot 1 + 6}\,e^x \\ &= \frac{1}{12}\,e^x. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 e^{-5x} + c_2 e^{-6x} + \frac{1}{12}\,e^x. \end{aligned}$$

This is the required solution. $\square$

Example 80. Solve: $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

Solution: Given differential equation can be written as:

$$F(D)x = \psi(x)$$

where $F(D) \equiv (D+2)(D-1)^2$ and $\psi(x) = e^{-2x} + 2\sinh x$. The auxiliary equation

of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad (m+2)(m-1)^2 = 0 \\ &\implies \quad m = -2, 1, 1. \end{aligned}$$

Therefore, one root of the auxiliary equation is real and the other two are real but equal, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)e^x.$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\, \psi(x) = \frac{1}{(D+2)(D-1)^2}[\, e^{-2x} + 2\sinh x] \\ &= \frac{1}{(D+2)(D-1)^2}\, e^{-2x} + 2\frac{1}{(D+2)(D-1)^2}\frac{e^x - e^{-x}}{2} \\ &= \frac{1}{(D+2)}\left[\frac{1}{(D-1)^2}\, e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{(D+2)}\, e^x\right] - \frac{1}{(D+2)(D-1)^2}\, e^{-x} \\ &= \frac{1}{(D+2)}\left[\frac{1}{(-2-1)^2}\, e^{-2x}\right] + \frac{1}{(D-1)^2}\left[\frac{1}{(1+2)}\, e^x\right] - \frac{1}{(-1+2)(-1-1)^2}\, e^{-x} \\ &= \frac{1}{9}\frac{1}{D+2}\, 1 \cdot e^{-2x} + \frac{1}{3}\frac{1}{(D-1)^2}\, 1 \cdot e^x - \frac{1}{4}\, e^{-x} \\ &= \frac{e^{-2x}}{9}\frac{1}{D-2+2}\, 1 + \frac{e^x}{3}\frac{1}{(D+1-1)^2}\, 1 - \frac{1}{4}\, e^{-x} \\ &= \frac{e^{-2x}}{9}\frac{1}{D}\, 1 + \frac{e^x}{3}\frac{1}{D^2}\, 1 - \frac{1}{4}\, e^{-x} \\ &= \frac{xe^{-2x}}{9} + \frac{x^2 e^x}{6} - \frac{1}{4}\, e^{-x}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 e^{-2x} + (c_2 + c_3 x)e^x + \frac{xe^{-2x}}{9} + \frac{x^2 e^x}{6} - \frac{1}{4}\, e^{-x}. \end{aligned}$$

This is the required solution. □

Example 81. Solve: $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$

Solution: Given differential equation can be written as:

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^3 - 3D^2 + 4D - 2$ and $\psi(x) = e^x + \cos x$. The auxiliary equation

of the above equation will be:

$$F(m) = 0 \implies m^3 - 3m^2 + 4m - 2 = 0.$$

Since the above equation is satisfied by $m = 1$, therefore, $(m-1)$ will be a factor, and so, we write:

$$\begin{aligned} & m^2(m-1) - 2m(m-1) + 2(m-1) = 0 \\ \implies & (m-1)(m^2 - 2m + 2) = 0 \\ \implies & m = 1, 1 \pm i. \end{aligned}$$

Therefore, the complementary function will be:

$$\text{C.F.} = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x).$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\psi(x) = \frac{1}{D^3 - 3D^2 + 4D - 2}[e^x + \cos x] \\ &= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x \\ &= \frac{1}{D-1}\left[\frac{1}{D^2 - 2D + 2}\right] e^x + \frac{1}{D \cdot D^2 - 3D^2 + 4D - 2} \cos x \\ &= \frac{1}{D-1}\left[\frac{1}{1^2 - 2 \cdot 1 + 2}\right] e^x + \frac{1}{D \cdot (-1^2) - 3(-1^2) + 4D - 2} \cos x \\ &= \frac{1}{D-1}(1 \cdot e^x) + \frac{1}{3D+1} \cos x \\ &= e^x \frac{1}{D+1-1} \cdot 1 + \frac{3D-1}{9D^2 - 1} \cos x \\ &= e^x \frac{1}{D} \cdot 1 + \frac{3D-1}{9(-1^2) - 1} \cos x \\ &= xe^x - \frac{1}{10}[3D(\cos x) - \cos x] \\ &= xe^x + \frac{1}{10}[3\sin x + \cos x]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 e^x + e^x(c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10}[3\sin x + \cos x]. \end{aligned}$$

This is the required solution. □

Example 82. Solve: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 4\cos^2 x.$

Solution: Given differential equation can be written as:

$$(D^2 + 3D + 2)y = 4\cos^2 x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2 + 3D + 2$ and $\psi(x) = 4\cos^2 x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 &\implies m^2 + 3m + 2 = 0 \\ &\implies m = -1, -2. \end{aligned}$$

Therefore, the roots are real and distinct, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}.$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2+3D+2}\,[4\cos^2 x] \\ &= 4 \cdot \frac{1}{D^2+3D+2}\,\cos^2 x \\ &= 2 \cdot \frac{1}{D^2+3D+2}\,[1+\cos(2x)] \\ &= 2\left[\frac{1}{D^2+3D+2}\,1 + \frac{1}{D^2+3D+2}\,\cos(2x)\right] \\ &= 2\left[\frac{1}{D^2+3D+2}\,e^0 + \frac{1}{-2^2+3D+2}\,\cos(2x)\right] \\ &= 2\left[\frac{1}{0+3\cdot 0+2}\,e^0 + \frac{1}{3D-2}\,\cos(2x)\right] \\ &= 2\left[\frac{1}{2} + \frac{3D+2}{9D^2-4}\,\cos(2x)\right] \\ &= 2\left[\frac{1}{2} + \frac{1}{9(-2^2)-4}\,(3D+2)\cos(2x)\right] \\ &= 1 - \frac{1}{20}\,[-6\sin(2x) + 2\cos(2x)]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 e^{-x} + c_2 e^{-2x} + 1 - \frac{1}{20}\,[-6\sin(2x) + 2\cos(2x)]. \end{aligned}$$

This is the required solution. □

Example 83. Solve: $\dfrac{d^2y}{dx^2} + 4y = e^x + \sin(2x)$.

Solution: Given differential equation can be written as:

$$(D^2+4)y = e^x + \sin(2x) \implies F(D)x = \psi(x)$$

where $F(D) \equiv D^2 + 4$ and $\psi(x) = e^x + \sin(2x)$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 + 4 = 0 \\ &\implies \quad m = \pm 2i. \end{aligned}$$

Therefore, the roots are complex, and so, the complementary function will be:

$$\text{C.F.} = e^{0 \cdot x}[c_1 \cos(2x) + c_2 \sin(2x)] = c_1 \cos(2x) + c_2 \sin(2x).$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2+4}\,[e^x + \sin(2x)] \\ &= \frac{1}{D^2+4}\,e^x + \frac{1}{D^2+4}\,\sin(2x) \\ &= \frac{1}{1^2+4}\,e^x - \frac{x}{2\cdot 2}\,\cos(2x) \qquad (\text{since } F(-a^2) = 0) \\ &= \frac{e^x}{5} - \frac{x}{4}\cos(2x). \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 \cos(2x) + c_2 \sin(2x) + \frac{e^x}{5} - \frac{x}{4}\cos(2x). \end{aligned}$$

This is the required solution. □

Example 84. Solve: $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x^2$.

Solution: Given differential equation can be written as:

$$(D^3 + 3D^2 + 2D)y = x^2 \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^3+3D^2+2D$ and $\psi(x) = x^2$. The auxiliary equation of the above

equation will be:

$$\begin{aligned} F(m)=0 &\implies m^3+3m^2+2m=0\\ &\implies m(m^2+3m+2)=0\\ &\implies m=0,-1,-2. \end{aligned}$$

Therefore, the roots are real and distinct, and so, the complementary function will be:

$$\text{C.F.}=c_1e^{0\cdot x}+c_2e^{-x}+c_2e^{-2x}=c_1+c_2e^{-x}+c_2e^{-2x}.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x)=\frac{1}{D^3+3D^2+2D}\,x^2\\ &= \frac{1}{2D}\cdot\frac{1}{1+\frac{D^2}{2}+\frac{3D}{2}}\,x^2\\ &= \frac{1}{2D}\left[1+\frac{D^2}{2}+\frac{3D}{2}\right]^{-1}x^2. \end{aligned}$$

Using the formula $(1+a)^{-1}=1-a+a^2-\cdots$ we get:

$$\begin{aligned} \text{P.I.} &= \frac{1}{2D}\left[1-\left(\frac{D^2}{2}+\frac{3D}{2}\right)+\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2+\cdots\right]x^2\\ &= \frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\frac{3D^3}{2}+\cdots\right]x^2\\ &= \frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]\\ &= \frac{x^3}{6}-\frac{3x^2}{4}+\frac{7x}{4}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.}+\text{P.I.}\\ \implies y &= c_1+c_2e^{-x}+c_2e^{-2}+\frac{x^3}{6}-\frac{3x^2}{4}+\frac{7x}{4}. \end{aligned}$$

This is the required solution. □

Example 85. Solve: $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=8(e^{2x}+\sin 2x+x^2)$.

Solution: Given differential equation can be written as:

$$(D^2-4D+4)y=x^2 \implies F(D)y=\psi(x)$$

where $F(D)\equiv D^2-4D+4$ and $\psi(x)=8(e^{2x}+\sin 2x+x^2)$. The auxiliary equation

of the above equation will be:

$$F(m)=0 \implies m^2-4m+4=0$$
$$\implies m=2,2.$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.}=(c_1+c_2x)e^{2x}.$$

The particular integral will be:

$$\begin{aligned}
\text{P.I.} &= \frac{1}{F(D)}\,\psi(x)=\frac{1}{D^2-4D+4}\,8(e^{2x}+\sin 2x+x^2)\\
&= 8\left[\frac{1}{D^2-4D+4}\,e^{2x}+\frac{1}{D^2-4D+4}\,\sin 2x+\frac{1}{D^2-4D+4}\,x^2\right]\\
&= 8\left[\frac{1}{D^2-4D+4}\,(1\cdot e^{2x})+\frac{1}{-2^2-4D+4}\,\sin 2x+\frac{1}{4}\cdot\frac{1}{1+\frac{D^2}{4}-D}\,x^2\right]\\
&= 8\left[e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}\cdot 1\right.\\
&\quad \left.-\frac{1}{4D}\,\sin 2x+\frac{1}{4}\left\{1+\left(\frac{D^2}{4}-D\right)\right\}^{-1}\,x^2\right]\\
&= 8\left[e^{2x}\frac{1}{D^2}\cdot 1+\frac{1}{8}\cos 2x+\frac{1}{4}\left\{1-\left(\frac{D^2}{4}-D\right)+\left(\frac{D^2}{4}-D\right)^2+\cdots\right\}\,x^2\right]\\
&= 8\left[\frac{x^2e^{2x}}{2}+\frac{\cos 2x}{8}+\frac{1}{4}\left\{x^2+2x+3/2\right\}\right].
\end{aligned}$$

The complete solution will be:

$$\begin{aligned}
y &= \text{C.F.}+\text{P.I.}\\
\implies y &= (c_1+c_2x)e^{2x}+4x^2e^{2x}+\cos 2x+2\left\{x^2+2x+3/2\right\}.
\end{aligned}$$

This is the required solution. □

Example 86. Solve: $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^{2x}+x^2+x.$

Solution: Given differential equation can be written as:

$$(D^3+2D^2+D)y=e^{2x}+x^2+x. \implies F(D)x=\psi(x)$$

where $F(D)\equiv D^3+2D^2+D$ and $\psi(x)=e^{2x}+x^2+x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m)=0 &\implies m^3+2m^2+m=0 \\ &\implies m(m^2+2m+1)=0 \\ &\implies m(m+1)^2=0 \\ &\implies m=0,-1,-1. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = c_1+(c_2+c_3x)e^{-x}.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^3+2D^2+D}\,[e^{2x}+x^2+x] \\ &= \frac{1}{D^3+2D^2+D}\,e^{2x}+\frac{1}{D^3+2D^2+D}(x^2+x) \\ &= \frac{1}{2^3+2\cdot 2^2+2}\,e^{2x}+\frac{1}{D^3+2D^2+D}(x^2+x) \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\left[\frac{1}{1+D^2+2D}(x^2+x)\right] \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\,[\{1+(D^2+2D)\}^{-1}(x^2+x)] \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\,[\{1-(D^2+2D)+(D^2+2D)^2-\cdots\}(x^2+x)] \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\,[\{1-2D+3D^2-\cdots\}(x^2+x)] \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\,[x^2+x-2(2x+1)+3\cdot 2] \\ &= \frac{e^{2x}}{18}+\frac{1}{D}\,[x^2-3x+4] \\ &= \frac{e^{2x}}{18}+\frac{x^3}{3}-\frac{3x^2}{2}+4x. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1+(c_2+c_3x)e^{-x}+\frac{e^{2x}}{18}+\frac{x^3}{3}-\frac{3x^2}{2}+4x. \end{aligned}$$

This is the required solution. □

Example 87. Solve: $(D^3-3D+2)y=540x^2e^{-x}$.

Solution: Given differential equation can be written as:

$$F(D)y = \psi(x)$$

where $F(D) \equiv D^3 - 3D + 2$ and $\psi(x) = 540x^2e^{-x}$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 &\implies m^3 - 3m + 2 = 0 \\ &\implies m^2(m-1) + m(m-1) - 2(m-1) = 0 \\ &\implies (m-1)(m^2+m-2) = 0 \\ &\implies m = 1, 1, -2. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = c_1e^{-2x} + (c_2 + c_3x)e^x.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^3 - 3D + 2}\,540x^2e^{-x} \\ &= 540\left[\frac{1}{D^3 - 3D + 2}\,x^2e^{-x}\right] \\ &= 540e^{-x}\left[\frac{1}{(D-1)^3 - 3(D-1) + 2}\,x^2\right] \\ &= 540e^{-x}\left[\frac{1}{D^3 - 3D^2 + 4}\,x^2\right] \\ &= \frac{540e^{-x}}{4}\left[\left\{1 + \frac{D^3 - 3D^2}{4}\right\}^{-1} x^2\right] \\ &= 135e^{-x}\left[\left\{1 - \frac{D^3}{4} + \frac{3D^2}{4} + \cdots\right\} x^2\right] \\ &= 135e^{-x}\left[x^2 + \frac{3}{2}\right]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1e^{-2x} + (c_2 + c_3x)e^x + 135e^{-x}\left[x^2 + \frac{3}{2}\right]. \end{aligned}$$

This is the required solution. □

Example 88. Solve: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x.$

Solution: Given differential equation can be written as:

$$(D^2 - 2D + 1)y = xe^x \sin x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2 - 2D + 1$ and $\psi(x) = xe^x \sin x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 - 2m + 1 = 0 \\ &\implies \quad (m-1)^2 = 0 \\ &\implies \quad m = 1, 1. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2 - 2D + 1}\,xe^x \sin x \\ &= \frac{1}{D^2 - 2D + 1}\,e^x \cdot x \sin x \\ &= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 1}\,x \sin x\right] \\ &= e^x \left[\frac{1}{D^2}\,x \sin x\right] \\ &= e^x \left[\frac{1}{D}\int x \sin x dx\right] \\ &= e^x \left[\frac{1}{D}\,(-x\cos x + \sin x)\right] \\ &= e^x \left[\int (-x\cos x + \sin x)dx\right] \\ &= -e^x(x \sin x + 2\cos x). \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x). \end{aligned}$$

This is the required solution. □

Example 89. Solve: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \sin x.$

Solution: Given differential equation can be written as:

$$(D^2 - 2D + 1)y = x \sin x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2 - 2D + 1$ and $\psi(x) = x \sin x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 - 2m + 1 = 0 \\ &\implies \quad (m-1)^2 = 0 \\ &\implies \quad m = 1, 1. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2 - 2D + 1}\,x \sin x \\ &= \frac{1}{D^2 - 2D + 1}\,x \cdot \sin x \\ &= x\,\frac{1}{(D^2 - 2D + 1)}\,\sin x + \frac{d}{dD}\left[\frac{1}{D^2 - 2D + 1}\right]\sin x \\ &= x\,\frac{1}{(D^2 - 2D + 1)}\,\sin x - \left[\frac{2D - 2}{(D^2 - 2D + 1)^2}\right]\sin x \\ &= x\,\frac{1}{(-1^2 - 2D + 1)}\,\sin x - \left[\frac{2D - 2}{(-1^2 - 2D + 1)^2}\right]\sin x \\ &= -x\,\frac{1}{2D}\,\sin x - \left[\frac{2D - 2}{4D^2}\right]\sin x \\ &= \frac{x \cos x}{2} - \left[\frac{D - 1}{2(-1^2)}\right]\sin x \\ &= \frac{x \cos x}{2} + \frac{\cos x - \sin x}{2}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= (c_1 + c_2 x)e^x + \frac{x \cos x}{2} + \frac{\cos x - \sin x}{2}. \end{aligned}$$

This is the required solution. □

Example 90. Solve: $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

Solution: Given differential equation can be written as:

$$F(D)y = \psi(x)$$

where $F(D) \equiv D^2 - 4D + 4$ and $\psi(x) = 8x^2 e^{2x} \sin 2x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 - 4m + 4 = 0 \\ &\implies \quad (m-2)^2 = 0 \\ &\implies \quad m = 2, 2. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^2 - 4D + 4}\,8x^2 e^{2x}\sin 2x \\ &= 8\left[\frac{1}{D^2 - 4D + 4}\,e^{2x}\cdot x^2 \sin 2x\right] \\ &= 8e^{2x}\left[\frac{1}{(D+2)^2 - 4(D+2) + 4}\,\cdot x^2 \sin 2x\right] \\ &= 8e^{2x}\left[\frac{1}{D^2}\,\cdot x^2 \sin 2x\right]. \end{aligned}$$

Operating $\frac{1}{D}$ we obtain:

$$\begin{aligned} \text{P.I.} &= 8e^{2x}\left[\frac{1}{D}\int x^2 \sin 2x dx\right] \\ &= 8e^{2x}\left[\frac{1}{D}\left(-\frac{x^2\cos 2x}{2} + \frac{2x\sin 2x}{4} + \frac{2\cdot\cos 2x}{8}\right)\right] \\ &= 8e^{2x}\left[\frac{1}{D}\left(-\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right)\right] \\ &= 8e^{2x}\left[-\int\left(\frac{x^2\cos 2x}{2}\right)dx + \int\left(\frac{x\sin 2x}{2}\right)dx + \int\left(\frac{\cos 2x}{4}\right)dx\right] \\ &= 8e^{2x}\left[-\frac{1}{2}\left(\frac{x^2\ \sin 2x}{2} + \frac{2x\cos 2x}{4} - \frac{2\cdot\sin 2x}{8}\right)\right. \\ &\quad \left. + \frac{1}{2}\left(-\frac{x\cos 2x}{2} + \frac{\sin x}{4}\right) + \left(\frac{\sin 2x}{8}\right)\right] \\ &= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= (c_1 + c_2 x)e^{2x} + e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]. \end{aligned}$$

This is the required solution. □

Example 91. Solve: $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=e^{e^x}$.

Solution: Given differential equation can be written as:

$$(D^2+3D+2)y=e^{e^x} \implies F(D)y=\psi(x)$$

where $F(D)\equiv D^2+3D+2$ and $\psi(x)=e^{e^x}$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m)=0 &\implies m^2+3m+2=0 \\ &\implies (m+1)(m+2)=0 \\ &\implies m=-1,-2. \end{aligned}$$

Therefore, the roots are real and distinct, and so, the complementary function will be:

$$\text{C.F.}=c_1e^{-x}+c_2e^{-2x}.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x)=\frac{1}{D^2+3D+2}\,e^{e^x} \\ &= \frac{1}{(D+1)(D+2)}\,e^{e^x} \\ &= \left[\frac{1}{D+1}-\frac{1}{D+2}\right]\,e^{e^x} \\ &= \frac{1}{D+1}\,e^{e^x}-\frac{1}{D+2}\,e^{e^x}. \end{aligned}$$

Applying the general formula $\dfrac{1}{D-m}f(x)=e^{mx}\displaystyle\int f(x)e^{-mx}dx$ we obtain:

$$\begin{aligned} \text{P.I.} &= e^{-x}\int e^{e^x}e^xdx-e^{-2x}\int e^{e^x}e^{2x}dx \\ &= e^{-x}\int e^tdt-e^{-2x}\int te^tdt \qquad (\text{put } e^x=t) \\ &= e^{-x}e^t-e^{-2x}(te^t-e^t) \\ &= e^{-x}e^{e^x}-e^{-2x}(e^xe^{e^x}-e^{e^x}) \\ &= e^{-2x}e^{e^x}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.}+\text{P.I.} \\ \implies y &= c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^x}. \end{aligned}$$

This is the required solution. □

Example 92. Solve: $(D^4+2D^2+1)y = x^2\cos x$.

Solution: Given differential equation can be written as:

$$F(D)y = \psi(x)$$

where $F(D) \equiv D^4+2D^2+1$ and $\psi(x) = x^2\cos x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m)=0 &\implies m^4+2m^2+1=0 \\ &\implies (m^2+1)^2 = 0 \\ &\implies m = \pm i. \pm i. \end{aligned}$$

Therefore, the roots are complex and repeated, and so, the complementary function will be:

$$\text{C.F.} = (c_1+c_2x)\cos x + (c_3+c_4x)\sin x.$$

The particular integral will be

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(x) = \frac{1}{D^4+2D^2+1}\,x^2\cos x \\ &= \text{Real part of } \left[\frac{1}{(D^2+1)^2}\,x^2e^{ix}\right] \\ &= \text{Real part of } \left[e^{ix}\left\{\frac{1}{((D+i)^2+1)^2}\,x^2\right\}\right] \\ &= \text{Real part of } \left[e^{ix}\left\{\frac{1}{(D^2+2iD)^2}\,x^2\right\}\right] \\ &= \text{Real part of } \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1+\frac{D}{2i}\right)^{-2}\,x^2\right\}\right] \\ &= \text{Real part of } \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1-2\frac{iD}{2}+3\left(\frac{iD}{2}\right)^2+\cdots\right)\,x^2\right\}\right] \\ &= \text{Real part of } \left[-\frac{e^{ix}}{4}\left\{\frac{1}{D^2}\left(x^2-2ix-\frac{3}{2}\right)\right\}\right]. \end{aligned}$$

Applying $\frac{1}{D^2}$ we obtain:

$$\begin{aligned} \text{P.I.} &= -\frac{1}{4}\text{Real part of } \left[e^{ix}\left\{\left(\frac{x^4}{12}-i\frac{x^3}{3}-\frac{3x^2}{4}\right)\right\}\right] \\ &= -\frac{1}{48}\text{Real part of } \left[(\cos x + i\sin x)\left\{\left(x^4+4ix^3-9x^2\right)\right\}\right] \\ &= -\frac{1}{48}\left[(x^4-9x^2)\cos x - 4x^3\sin x\right]. \end{aligned}$$

The complete solution will be:

$$y = \text{C.F.} + \text{P.I.}$$
$$\implies y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{1}{48}\left[(x^4 - 9x^2)\cos x - 4x^3 \sin x\right].$$

This is the required solution. □

Exercise (Assignment)

(Q.1) Solve: $(D-1)^2(D-3)^3y = e^{3x}$

Ans. $y = (c_1 + c_2x)e^x + (c_3 + c_4x + c_5x^2)e^{3x} + \frac{x^3e^{3x}}{24}$.

(Q.2) Solve: $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$.

Ans. $y = c_1e^{-x} + e^{x/2}\left[c_2\cos\frac{\sqrt{3}x}{2} + c_3\sin\frac{\sqrt{3}x}{2}\right] + 3 + \frac{5}{9}e^{2x} + \frac{xe^{-x}}{3}$.

(Q.3) Solve: $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos(2x)$.

Ans. $y = c_1e^x + (c_2 + c_3x)e^{-x} - \frac{1}{25}[\cos(2x) + 2\sin(2x)]$.

(Q.4) Solve: $\frac{d^4y}{dx^4} - m^4y = \cos(mx)$.

Ans. $y = c_1e^{mx} + c_2e^{-mx} + c_3\cos(mx) + c_4\sin(mx) - \frac{x}{4m^3}\sin(mx)$.

(Q.5) Solve: $\frac{d^2y}{dx^2} + 4y = x^2 + \cos^2 x$.

Ans. $y = c_1\cos(2x) + c_2\sin(2x) + \frac{x^2}{4} + \frac{x}{8}\sin(2x)$.

(Q.6) Solve: $\frac{d^4y}{dx^4} - a^4y = x^4$.

Ans. $y = c_1e^{-ax} + c_2e^{ax} + c_3\cos(ax) + c_4\sin(ax) - \frac{1}{a^4}(x^4 + 24/a^4)$.

(Q.7) Solve: $(D^2 + 2D + 4)y = e^x\sin 2x$.

Ans. $y = e^{-x}\left[c_1\cos\left(\sqrt{3}x\right) + c_2\sin\left(\sqrt{3}x\right)\right] + \frac{e^x}{73}(3\sin 2x - 8\cos 2x)$.

(Q.8) Solve: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$.

Ans. $y = c_1e^{-2x} + c_2e^{-3x} - \frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x)$.

(Q.9) Solve: $\frac{d^2y}{dx^2} - 4y = x\sinh x$.

Ans. $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

(Q.10) Solve: $(D^2+1)(D^2-1)y = e^{2x}x + x^2$.

Ans. $y = c_1e^x + c_2e^{-x} + c_2\cos x + c_4\sin x + \frac{e^{2x}}{255}(15x-32) + x^2$.

(Q.11) Solve: $\dfrac{d^2y}{dx^2} + 4y = x\sin x$.

Ans. $y = c_1\cos(2x) + c_2\sin(2x) + \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

(Q.12) Solve: $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2xe^{3x} + 3e^x\cos 2x$.

Ans. $y = c_1e^x + c_2e^{3x} + \frac{1}{2}xe^{3x}(x-1) - \frac{3}{8}e^x(\sin 2x - \cos 2x)$.

(Q.13) Solve: $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$.

Ans. $y = c_1\cos ax + c_2\sin ax - \frac{x}{2x}\cos ax$.

(Q.14) Solve: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = \dfrac{4e^x}{x^2}$.

Ans. $y = (c_1 + c_2x)e^x - 4e^x\ln(x)$.

(Q.15) Solve: $(D^2+2D+1)y = x\cos x$.

Ans. $y = (c_1+c_2x)e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$.

(Q.16) Solve: $(D^2-3D+2)y = \sin(e^x)$.

Ans. $y = c_1e^x + c_2e^{2x} - e^{2x}\sin(e^x)$.

Method of variation of parameters

This method is applied on the differential equations of the form:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \psi(x)$$

where p, q and $\psi(x)$ are the functions of x. Suppose, the complementary function of this equation is

$$\text{C.F.} = c_1y_1 + c_2y_2.$$

Then, the particular integral of this equation is given by:

$$\text{P.I.} = -y_1\int \frac{y_2\psi(x)}{W}dx + y_2\int \frac{y_1\psi(x)}{W}dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$ is the Wronskian of y_1 and y_2.

Example 93. Solve: $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$.

Solution: Given differential equation can be written as:

$$(D^2+4)y = \tan 2x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2+4$ and $\psi(x) = \tan 2x$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 &\implies m^2+4=0 \\ &\implies m = \pm 2i. \end{aligned}$$

Therefore, the roots are complex, and so, the complementary function will be:

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2.$$

Here $y_1 = \cos 2x,\ y_2 = \sin 2x$. Now the Wornskian of y_1 and y_2 will be:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2.$$

Therefore, the particular integral will be:

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 \psi(x)}{W} dx + y_2 \int \frac{y_1 \psi(x)}{W} dx \\ &= -\cos 2x \int \frac{\sin 2x \ \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \ \tan 2x}{2} dx \\ &= -\frac{\cos 2x}{2} \int (1-\cos^2 2x) \ \sec 2x dx + \frac{\sin 2x}{2} \int \sin 2x dx \\ &= -\frac{\cos 2x}{2} \int (\sec 2x - \cos 2x) \ dx - \frac{\sin 2x \cos 2x}{4} \\ &= -\frac{\cos 2x}{4} \left[\ln(\sec 2x + \tan 2x) - \sin 2x\right] - \frac{\sin 2x \cos 2x}{4} \\ &= -\frac{1}{4} \cos 2x \cdot \ln(\sec 2x + \tan 2x). \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \cdot \ln(\sec 2x + \tan 2x). \end{aligned}$$

This is the required solution. □

Example 94. Solve: $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax.$

Solution: Given differential equation can be written as:

$$(D^2+a^2)y = \sec 2x \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2 + a^2$ and $\psi(x) = \sec ax$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^2 + a^2 = 0 \\ &\implies \quad m = \pm ia. \end{aligned}$$

Therefore, the roots are complex, and so, the complementary function will be:

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax = c_1 y_1 + c_2 y_2.$$

Here $y_1 = \cos ax,\ y_2 = \sin ax$. Now the Wornskian of y_1 and y_2 will be:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a.$$

Therefore, the particular integral will be:

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 \psi(x)}{W} dx + y_2 \int \frac{y_1 \psi(x)}{W} dx \\ &= -\cos ax \int \frac{\sin ax \ \sec ax}{a} dx + \sin ax \int \frac{\cos ax \ \sec ax}{a} dx \\ &= -\frac{\cos ax}{a} \int \tan ax dx + \frac{\sin ax}{a} \int dx \\ &= -\frac{\cos ax}{a^2} [\ln(\sec ax)] + \frac{x \sin ax}{a} \\ &= \frac{1}{a^2} \cos ax \cdot \ln(\cos ax) + \frac{1}{a} x \sin ax. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{a^2} \cos ax \cdot \ln(\cos ax) + \frac{1}{a} x \sin ax. \end{aligned}$$

This is the required solution. □

Example 95. Solve: $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$.

Solution: Given differential equation can be written as:

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2} \implies F(D)y = \psi(x)$$

where $F(D) \equiv D^2 - 6D + 9$ and $\psi(x) = \dfrac{e^{3x}}{x^2}$. The auxiliary equation of the above

equation will be:

$$\begin{aligned} F(m)=0 &\implies m^2-6m+9=0 \\ &\implies (m-3)^2=0 \\ &\implies m=3,3. \end{aligned}$$

Therefore, the roots are real and equal, and so, the complementary function will be:

$$\text{C.F.} = c_1e^{3x} + c_2xe^{3x} = c_1y_1 + c_2y_2.$$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$. Now the Wornskian of y_1 and y_2 will be:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix} = e^{6x}.$$

Therefore, the particular integral will be:

$$\begin{aligned} \text{P.I.} &= -y_1\int \frac{y_2\psi(x)}{W}dx + y_2\int \frac{y_1\psi(x)}{W}dx \\ &= -e^{3x}\int \frac{xe^{3x}\ e^{3x}}{x^2e^{6x}}dx + xe^{3x}\int \frac{e^{3x}e^{3x}}{x^2}e^{6x}dx \\ &= -e^{3x}\int \frac{1}{x}dx + xe^{3x}\int \frac{1}{x^2}dx \\ &= -e^{3x}(\ln x + 1). \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies y &= c_1e^{3x} + c_2xe^{3x} - e^{3x}(\ln x + 1). \end{aligned}$$

This is the required solution. □

Exercise (Assignment)

(Q.1) Solve by the method of variation of parameter: $\dfrac{d^2y}{dx^2} + y = \tan x$

Ans. $y = c_1\cos x + c_2\sin x - \cos x\cdot\ln(\sec x + \tan x)$.

(Q.2) Solve by the method of variation of parameter: $\dfrac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$

Ans. $y = (c_1 - x/a)\cos ax + [c_2 + (1/a^2)\ln(\sin ax)]\sin ax$.

(Q.3) Solve by the method of variation of parameter: $y'' - 2y' + y = e^x\ln(x)$.

Ans. $y = c_1e^x + c_2xe^x + \frac{1}{4}x^2e^x(2\ln x - 3)$.

Problems on operator method

Example 96. Prove the following result:

If $(D - m_1)(D - m_2)y = 0$, then $y = c_1 e^{m_1 x} + c_2 e^{m_2} x$, where $m_1 \neq m_2$.

Solution: Putting $(D - m_2)y = z$ in the given equation we get:

$$\begin{aligned}(D - m_1)z = 0 \quad &\Longrightarrow \quad \frac{dz}{dx} - m_1 z = 0 \Longrightarrow \frac{dz}{dx} = m_1 z \\ &\Longrightarrow \quad \frac{dz}{z} = m_1 dx \\ &\Longrightarrow \quad \ln(z) = \ln(c) + m_1 x \\ &\Longrightarrow \quad z = ce^{m_1 x}.\end{aligned}$$

Putting this value of z in $(D - m_2)y = z$ we get:

$$\begin{aligned}(D - m_2)y = z \quad &\Longrightarrow \quad (D - m_2)y = ce^{m_1 x} \\ &\Longrightarrow \quad \frac{dy}{dx} - m_2 y = ce^{m_1 x}.\end{aligned}$$

The above equation is linear in y, and I.F.$= e^{\int -m_2 dx} = e^{-m_2 x}$ and the solution of it will be:

$$\begin{aligned}ye^{-m_2 x} = c_2 + \int ce^{m_1 x} e^{-m_2 x} dx \quad &\Longrightarrow \quad ye^{-m_2 x} = c_2 + \frac{c}{m_1 - m_2} e^{(m_1 - m_2)x} \\ &\Longrightarrow \quad y = c_2 e^{m_2 x} + c_1 e^{m_1 x}\end{aligned}$$

where $c_1 = \dfrac{c}{m_1 - m_2}$. □

Example 97. Prove the following result:

If $(D - m)^2 y = 0$, then $y = (c_1 + c_2 x)e^{mx}$.

Solution: Putting $(D - m)y = z$ in the given equation we get:

$$\begin{aligned}(D - m)z = 0 \quad &\Longrightarrow \quad \frac{dz}{dx} - mz = 0 \Longrightarrow \frac{dz}{dx} = mz \\ &\Longrightarrow \quad \ln(z) = \ln(c_1) + mx \\ &\Longrightarrow \quad z = c_1 e^{mx}.\end{aligned}$$

Putting this value of z in $(D - m)y = z$ we get:

$$\begin{aligned}(D - m)y = z \quad &\Longrightarrow \quad (D - m)y = c_1 e^{mx} \\ &\Longrightarrow \quad \frac{dy}{dx} - my = c_1 e^{mx}.\end{aligned}$$

The above equation is linear in y, and I.F.$= e^{\int -m dx} = e^{-mx}$ and the solution of it will be:

$$ye^{-mx} = c_2 + \int c_1 e^{mx} e^{-mx} dx \implies ye^{-mx} = c_2 + c_1 x$$
$$\implies y = (c_2 + c_1 x)e^{mx}. \qquad \square$$

Example 98. Prove the following result:

If $[D - (\alpha + i\beta)][D - (\alpha - i\beta)]y = 0$, then $y = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

Solution: Put $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ in Example 96 and then use the Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$.

Example 99. Solve by operator method: $(x+3)y'' - (2x+7)y' + 2y = (x+3)^2 e^x$.

Solution: The given equation can be written as:

$$\left[(x+3)D^2 - (2x+7)D + 2\right] y = (x+3)^2 e^x.$$

To apply the operator method, we have to factorize the expression

$$(x+3)D^2 - (2x+7)D + 2.$$

Then the above equation can be written as:

$$\begin{aligned} \left[(x+3)D^2 - (2x+6)D - D + 2\right] y &= (x+3)^2 e^x \\ \implies \left[(x+3)D(D-2) - (D-2)\right] y &= (x+3)^2 e^x \\ \implies \left[(x+3)D - 1\right](D-2)y &= (x+3)^2 e^x. \end{aligned}$$

Putting $(D-2)y = z$ we obtain:

$$\begin{aligned} \left[(x+3)D - 1\right] z &= (x+3)^2 e^x \\ \implies (x+3)\frac{dz}{dx} - z = (x+3)^2 e^x & \\ \implies \frac{dz}{dx} - \frac{1}{x+3} z = (x+3)e^x. & \end{aligned}$$

It is linear differential equation in z. Now:

$$\text{I.F.} = e^{\int -\frac{1}{x+3} dx} = \frac{1}{x+3}.$$

The solution of the above equation will be:

$$\begin{aligned} \frac{z}{x+3} &= c_1 + \int (x+3)e^x \cdot \frac{1}{x+3} \\ \implies z &= c_1(x+3) + e^x(x+3). \end{aligned}$$

Putting $z = (D-2)y = \frac{dy}{dx} - 2y$ we get

$$\frac{dy}{dx} - 2y = c_1(x+3) + e^x(x+3).$$

It is again linear in y, and

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}.$$

The solution will be:

$$\begin{aligned} ye^{-2x} &= c_2 + \int [c_1(x+3) + e^x(x+3)]\, e^{-2x} dx \\ \implies ye^{-2x} &= c_2 + \int \left[c_1(x+3)e^{-2x} + e^{-x}(x+3)\right] dx \\ \implies ye^{-2x} &= c_2 - \frac{1}{2}c_2(x+3)e^{-2x} - \frac{1}{4}c_1 e^{-2x} - (x+3)e^{-x} - e^{-x} \\ \implies y &= c_2 e^{2x} - \frac{1}{2}c_2(x+3) - \frac{1}{4}c_1 - (x+3)e^x - e^x. \end{aligned}$$

This is the required solution. □

Equations reducible to linear equation with constant coefficients (Cauchy homogeneous linear equation)

A differential equation of the following form:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = \phi(x). \quad (7)$$

is called Cauchy homogeneous linear equation or homogeneous linear equation. To solve this type of equations we use the substitution $x = e^z$, so that $\frac{dx}{dz} = e^z = x$. Hence, if we denote $D \equiv \frac{d}{dz}$ we have:

$$x\frac{dy}{dx} = x\frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz} = Dy.$$

On comparing we get:

$$x\frac{d}{dx} \equiv D.$$

Similarly, we can show that:

$$x^2\frac{d^2}{dx^2} \equiv D(D-1), x^3\frac{d^3}{dx^3} \equiv D(D-1)(D-2) \text{ and so on.}$$

Now equation (7) reduces into the equation with costant coefficients and can be solved by the methods we have already discussed.

Example 100. Solve: $x^4\dfrac{d^3y}{dx^3}+2x^3\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=1.$

Solution: Given differential equation is:

$$x^4\frac{d^3y}{dx^3}+2x^3\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=1.$$

On dividing by x we get:

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\frac{1}{x}.$$

The above equation is Cauchy homogeneous linear equation, hence putting

$$x=e^z, x\frac{d}{dx}\equiv D, x^2\frac{d^2}{dx^2}\equiv D(D-1),\text{ and}$$

$$x^3\frac{d^3}{dx^3}\equiv D(D-1)(D-2)$$

(where $D\equiv\frac{d}{dz}$) we get:

$$\begin{aligned}
& D(D-1)(D-2)y+2D(D-1)y-Dy+y=\frac{1}{x}\\
\implies\ & \left[D\left(D^2-3D+2\right)+2D^2-2D-D+1\right]y=e^{-z}\\
\implies\ & \left(D^3-D^2-D+1\right)y=e^{-z}\\
\implies\ & F(D)y=\psi(z)
\end{aligned}$$

where:

$$F(D)\equiv D^3-D^2-D+1\text{ and}$$
$$\psi(z)=e^{-z}.$$

The auxiliary equation of the above equation will be:

$$\begin{aligned}
& F(m)=0\\
\implies\ & m^3-m^2-m+1=0\\
\implies\ & m^2(m-1)-(m-1)=0\\
\implies\ & (m-1)(m^2-1)=0\\
\implies\ & (m-1)(m-1)(m+1)=0\\
\implies\ & m=-1,1,1.
\end{aligned}$$

Therefore, all the roots are real, one root is distinct and other are equal, and so, the complementary function will be:

$$\text{C.F.}=c_1e^{-z}+(c_2+c_3z)e^z.$$

The particular integral will be:

$$\begin{aligned}
\text{P.I.} &= \frac{1}{F(D)}\,\psi(z) = \frac{1}{D^3 - D^2 - D + 1}\,e^{-z}\\
&= \frac{1}{(D+1)(D-1)^2}\,e^{-z}\\
&= \frac{1}{(D+1)}\left[\frac{1}{(D-1)^2}\,e^{-z}\right]\\
&= \frac{1}{(D+1)}\left[\frac{1}{(-1-1)^2}\,e^{-z}\right]\\
&= \frac{1}{4}\frac{1}{(D+1)}\,e^{-z}\cdot 1\\
&= \frac{e^{-z}}{4}\frac{1}{(D-1+1)}\,1\\
&= \frac{e^{-z}}{4}\frac{1}{D}\,1\\
&= \frac{ze^{-z}}{4}.
\end{aligned}$$

The complete solution will be:

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.}\\
\implies & y = c_1e^{-z} + (c_2 + c_3 z)e^{z} + \frac{ze^{-z}}{4}\\
\implies & y = c_1x^{-1} + (c_2 + c_3\ln(x))x + \frac{1}{4}x^{-1}\ln(x).
\end{aligned}$$

This is the required solution. □

Example 101. Solve: $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 13\cos(\ln x)$.

Solution: Given differential equation is:

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\cos(\ln x).$$

The above equation is Cauchy homogeneous linear equation, hence putting $x = e^z$, $x\,\frac{d}{dx} \equiv D, x^2\,\frac{d^2}{dx^2} \equiv D(D-1)$, and $x^3\,\frac{d^3}{dx^3} \equiv D(D-1)(D-2)$ (where $D \equiv \frac{d}{dz}$) we get:

$$\begin{aligned}
& D(D-1)(D-2)y + 3D(D-1)y + Dy + 8y = 13\cos z\\
\implies & \left(D^3 + 8\right)y = 13\cos z\\
\implies & F(D)y = \psi(z)
\end{aligned}$$

where $F(D) \equiv D^3 + 8$ and $\psi(z) = 13\cos z$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \quad &\implies \quad m^3 + 8 = 0 \\ &\implies \quad (m+2)(m^2 - 2m + 4) = 0 \\ &\implies \quad m = -2, \frac{2 \pm \sqrt{4 - 16}}{2} \\ &\implies \quad m = -2, 1 \pm i\sqrt{3}. \end{aligned}$$

Therefore, one root is real and two roots are complex, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-2z} + e^z \left[c_2 \cos(\sqrt{3}z) + c_3 \sin(\sqrt{3}z) \right].$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} \quad &= \quad \frac{1}{F(D)} \psi(z) = \frac{1}{D^3 + 8} 13 \cos z \\ &= \quad 13 \cdot \frac{1}{D \cdot D^2 + 8} \cos z \\ &= \quad 13 \cdot \frac{1}{-1^2 D + 8)} \cos z \\ &= \quad 13 \cdot \frac{8 + D}{64 - D^2} \cos z \\ &= \quad 13 \cdot \frac{8 + D}{64 - (-1^2)} \cos z \\ &= \quad \frac{1}{5} \left[8 \cos z - \sin z \right]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} & y = \text{C.F.} + \text{P.I.} \\ \implies \quad & y = c_1 e^{-2z} + e^z \left[c_2 \cos(\sqrt{3}z) + c_3 \sin(\sqrt{3}z) \right] + \frac{1}{5} \left[8 \cos z - \sin z \right] \\ \implies \quad & y = c_1 x^{-2} + x \left[c_2 \cos(\sqrt{3} \ln x) + c_3 \sin(\sqrt{3} \ln x) \right] + \frac{1}{5} \left[8 \cos(\ln x) - \sin(\ln x) \right]. \end{aligned}$$

This is the required solution. □

Example 102. Solve: $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = \dfrac{x^3}{1 + x^2}$.

Solution: Given differential equation is:

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = \frac{x^3}{1 + x^2}.$$

The above equation is Cauchy homogeneous linear equation, hence putting

$$x = e^z, x\frac{d}{dx} \equiv D, x^2\frac{d^2}{dx^2} \equiv D(D-1), \text{ and}$$

$$x^3\frac{d^3}{dx^3} \equiv D(D-1)(D-2)$$

(where $D \equiv \frac{d}{dz}$) we get:

$$\begin{aligned} & D(D-1)y + Dy - y = \frac{x^3}{1+x^2} \\ \implies & \left(D^2-1\right)y = \frac{x^3}{1+x^2} \\ \implies & F(D)y = \psi(z) \end{aligned}$$

where $F(D) \equiv D^2 - 1$ and $\psi(z) = \dfrac{x^3}{1+x^2}$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \implies & m^2 - 1 = 0 \\ \implies & (m-1)(m+1) = 0 \\ \implies & m = 1, -1. \end{aligned}$$

Therefore, one root are real distinct, and so, the complementary function will be:

$$\text{C.F.} = c_1e^z + c_2e^{-z}.$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\psi(z) = \frac{1}{D^2-1}\frac{x^3}{1+x^2} \\ &= \frac{1}{(D-1)(D+1)}\frac{e^{3z}}{1+e^{2z}} \\ &= \frac{1}{2}\cdot\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\frac{e^{3z}}{1+e^{2z}} \\ &= \frac{1}{2}\cdot\left[\frac{1}{D-1}\frac{e^{3z}}{1+e^{2z}} - \frac{1}{D+1}\frac{e^{3z}}{1+e^{2z}}\right] \\ &= \frac{1}{2}\cdot\left[e^z\int e^{-z}\frac{e^{3z}}{1+e^{2z}}dz - e^{-z}\int e^z\frac{e^{3z}}{1+e^{2z}}dz\right] \\ &= \frac{1}{2}\cdot\left[e^z\int \frac{e^{2z}}{1+e^{2z}}dz - e^{-z}\int \frac{e^{4z}}{1+e^{2z}}dz\right]. \end{aligned}$$

Putting $e^{2z} = u$, i.e., $e^{2z}dz = \frac{1}{2}du$, we get:

$$\begin{aligned} \text{P.I.} &= \frac{1}{4}\cdot\left[e^{z}\int\frac{1}{1+u}du - e^{-z}\int\frac{u}{1+u}du\right] \\ &= \frac{1}{4}\cdot\left[e^{z}\ln(1+u) - e^{-z}\left\{u-\ln(1+u)\right\}\right] \\ &= \frac{1}{4}\cdot\left[e^{z}\ln\left(1+e^{2z}\right) - e^{-z}\left\{e^{2z}-\ln\left(1+e^{2z}\right)\right\}\right] \\ &= \frac{1}{4}\cdot\left[e^{z}\ln\left(1+e^{2z}\right) - e^{z} + e^{-z}\ln\left(1+e^{2z}\right)\right]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies & y = c_1e^{z} + C_2e^{-z} + \frac{1}{4}\cdot\left[e^{z}\ln\left(1+e^{2z}\right) - e^{z} + e^{-z}\ln\left(1+e^{2z}\right)\right] \\ \implies & y = c_1x + c_2x^{-1} + \frac{1}{4}\cdot\left[x\ln\left(1+x^2\right) - x + x^{-1}\ln\left(1+x^2\right)\right]. \end{aligned}$$

This is the required solution. □

Example 103. Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\ln x$.

Solution: Given differential equation is

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\ln x.$$

The above equation is Cauchy homogeneous linear equation, hence putting $x = e^z$, $x\frac{d}{dx} \equiv D, x^2\frac{d^2}{dx^2} \equiv D(D-1)$, (where $D \equiv \frac{d}{dz}$) we get:

$$\begin{aligned} & D(D-1)y - 2Dy - 4y = x^2 + 2\ln x \\ \implies & \left(D^2 - 3D - 4\right)y = e^{2z} + 2z \\ \implies & F(D)y = \psi(z) \end{aligned}$$

where $F(D) \equiv D^2 - 3D - 4$ and $\psi(z) = e^{2z} + 2z$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 &\implies m^2 - 3m - 4 = 0 \\ &\implies (m+1)(m-4) = 0 \\ &\implies m = -1, 4. \end{aligned}$$

Therefore, one root are real distinct, and so, the complementary function will be:

$$\text{C.F.} = c_1e^{-z} + c_2e^{4z}.$$

The particular integral will be:

$$\begin{aligned}
\text{P.I.} &= \frac{1}{F(D)}\,\psi(z) = \frac{1}{D^2-3D-4}\,\left[e^{2z}+2z\right] \\
&= \frac{1}{D^2-3D-4}\,e^{2z} + 2\cdot\frac{1}{D^2-3D-4}\,z \\
&= \frac{1}{2^2-3\cdot 2-4}\,e^{2z} - \frac{1}{2}\cdot\frac{1}{1-\dfrac{D^2-3D}{4}}\,z \\
&= -\frac{1}{6}\,e^{2z} - \frac{1}{2}\cdot\left(1+\frac{D^2-3D}{4}+\cdots\right)\,z \\
&= -\frac{1}{6}\,e^{2z} - \frac{1}{2}\cdot\left(z+\frac{0-3}{4}\right) \\
&= -\frac{1}{6}\,e^{2z} - \frac{1}{2}\cdot\left(z-\frac{3}{4}\right).
\end{aligned}$$

The complete solution will be:

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.} \\
\Longrightarrow\ & y = c_1 e^{-z} + c_2 e^{4z} - \frac{1}{6}\,e^{2z} - \frac{1}{2}\cdot\left(z-\frac{3}{4}\right) \\
\Longrightarrow\ & y = c_1 x^{-1} + c_2 x^4 - \frac{1}{6}\,x^2 - \frac{1}{2}\cdot\left(\ln x - \frac{3}{4}\right).
\end{aligned}$$

This is the required solution. □

Example 104. Solve: $x^2\,\dfrac{d^2y}{dx^2} - 3x\,\dfrac{dy}{dx} + 4y = 3x^2$.

Solution: Given differential equation is

$$x^2\,\frac{d^2y}{dx^2} - 3x\,\frac{dy}{dx} + 4y = 3x^2.$$

The above equation is Cauchy homogeneous linear equation, hence putting $x = e^z$, $x\,\frac{d}{dx} \equiv D, x^2\,\frac{d^2}{dx^2} \equiv D(D-1)$, (where $D \equiv \frac{d}{dz}$) we get:

$$\begin{aligned}
& D(D-1)y - 3Dy + 4y = 3x^2 \\
\Longrightarrow\ & \left(D^2-4D+4\right)\,y = 3e^{2z} \\
\Longrightarrow\ & F(D)y = \psi(z)
\end{aligned}$$

where $F(D) \equiv D^2-4D+4$ and $\psi(z) = 3e^{2z}$. The auxiliary equation of the above

equation will be:

$$\begin{aligned} F(m)=0 &\implies m^2-4m+4=0 \\ &\implies (m-2)^2=0 \\ &\implies m=2,2. \end{aligned}$$

Therefore, one root are real equal, and so, the complementary function will be:

$$\text{C.F.}=(c_1+c_2z)e^{2z}.$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(z)=\frac{1}{D^2-4D+4}\,3e^{2z} \\ &= 3\cdot\frac{1}{(D-2)^2}\,e^{2z}\cdot 1 \\ &= 3e^{2z}\cdot\frac{1}{(D+2-2)^2}\,1 \\ &= 3e^{2z}\cdot\frac{1}{D^2}\,1 \\ &= \frac{3}{2}z^2e^{2z}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.}+\text{P.I.} \\ &\implies y=(c_1+c_2z)e^{2z}+\frac{3}{2}z^2e^{2z} \\ &\implies y=(c_1+c_2\ln x)x^2+\frac{3x^2}{2}(\ln x)^2. \end{aligned}$$

This is the required solution. □

Example 105. Solve: $x^3\,\dfrac{d^3y}{dx^3}+2x^2\,\dfrac{d^2y}{dx^2}+2y=10\left(x+\dfrac{1}{x}\right)$.

Solution: Given differential equation is

$$x^3\,\frac{d^3y}{dx^3}+2x^2\,\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right).$$

The above equation is Cauchy homogeneous linear equation, hence putting $x=e^z$, $x\,\frac{d}{dx}\equiv D, x^2\,\frac{d^2}{dx^2}\equiv D(D-1)$, and $x^3\,\frac{d^3}{dx^3}\equiv D(D-1)(D-2)$ (where $D\equiv\frac{d}{dz}$) we get:

$$\begin{aligned} & D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left(x + \frac{1}{x}\right) \\ \implies & \left(D^3 - D^2 + 2\right) y = 10\left(e^z + e^{-z}\right) \\ \implies & (D+1)\left(D^2 - 2D + 2\right) y = 10\left(e^z + e^{-z}\right) \\ \implies & F(D)y = \psi(z) \end{aligned}$$

where $F(D) \equiv (D+1)\left(D^2 - 2D + 2\right)$ and $\psi(z) = 10\left(e^z + e^{-z}\right)$. The auxiliary equation of the above equation will be:

$$\begin{aligned} F(m) = 0 \implies & (m+1)\left(m^2 - 2m + 2\right) = 0 \\ \implies & m = -1, 1 \pm i. \end{aligned}$$

Therefore, one root is real and two roots are complex, and so, the complementary function will be:

$$\text{C.F.} = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z).$$

The particular integral will be:

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D)}\,\psi(z) = \frac{1}{(D+1)\left(D^2 - 2D + 2\right)}\,10\left(e^z + e^{-z}\right) \\ &= 10 \cdot \frac{1}{(D+1)\left(D^2 - 2D + 2\right)}\,e^z + 10 \cdot \frac{1}{(D+1)\left(D^2 - 2D + 2\right)}\,e^{-z} \\ &= 10 \cdot \frac{1}{(1+1)\left(1^2 - 2\cdot 1 + 2\right)}\,e^z + 10 \cdot \frac{1}{(D+1)\left((-1)^2 - 2\cdot(-1) + 2\right)}\,e^{-z} \\ &= 5e^z + 2 \cdot \frac{1}{D+1}\,e^{-z} \cdot 1 \\ &= 5e^z + 2e^{-z} \cdot \frac{1}{D-1+1} \cdot 1 \\ &= 5e^z + 2e^{-z} \cdot \frac{1}{D} \cdot 1 \\ &= 5e^z + 2ze^{-z}. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ \implies & y = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z} \\ \implies & y = c_1 x^{-1} + x\left[c_2 \cos(\ln x) + c_3 \sin(\ln x)\right] + 5x + 2x^{-1} \ln x. \end{aligned}$$

This is the required solution. □

Example 106. Solve: $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$.

Solution: Given differential equation is

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}.$$

The above equation is Cauchy homogeneous linear equation, hence putting

$$x=e^z, x\frac{d}{dx}\equiv D, x^2\frac{d^2}{dx^2}\equiv D(D-1)$$

(where $D\equiv\frac{d}{dz}$) we get:

$$\begin{aligned}
&D(D-1)y+3D+y=\frac{1}{(1-x)^2}\\
\implies\quad &\left(D^2+2D+1\right)y=\frac{1}{(1-e^z)^2}\\
\implies\quad &F(D)y=\psi(z)
\end{aligned}$$

where

$$F(D)\equiv D^2+2D+1 \text{ and } \psi(z)=\frac{1}{(1-e^z)^2}.$$

The auxiliary equation of the above equation will be:

$$\begin{aligned}
F(m)=0 \quad&\implies\quad m^2+2m+1=0\\
&\implies\quad m=-1,-1.
\end{aligned}$$

Therefore, roots are real and equal, and so, the complementary function will be:

$$\text{C.F.}=(c_1+c_2z)e^{-z}.$$

The particular integral will be:

$$\begin{aligned}
\text{P.I.}\quad &= \frac{1}{F(D)}\psi(z)=\frac{1}{D^2+2D+1}\frac{1}{(1-e^z)^2}\\
&= \frac{1}{D^2+2D+1}\frac{1}{(1-e^z)^2}\\
&= \frac{1}{D+1}\cdot\frac{1}{D+1}\frac{1}{(1-e^z)^2}\\
&= \frac{1}{D+1}\left[e^{-z}\int e^z\frac{1}{(1-e^z)^2}\right]dz\\
&= \frac{1}{D+1}\left[\frac{e^{-z}}{1-e^z}\right]\\
&= e^{-z}\int e^z\cdot\frac{e^{-z}}{1-e^z}dz.
\end{aligned}$$

Putting $e^z = u$ we get:

$$\begin{aligned} \text{P.I.} &= e^{-z}\int \frac{1}{u(1-u)}du = e^{-z}\int\left[\frac{1}{u}+\frac{1}{1-u}\right]du \\ &= e^{-z}\left[\ln u - \ln(1-u)\right] \\ &= e^{-z}\left[z-\ln(1-e^z)\right]. \end{aligned}$$

The complete solution will be:

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &\Longrightarrow y = (c_1+c_2 z)e^{-z} + e^{-z}\left[z-\ln(1-e^z)\right] \\ &\Longrightarrow y = (c_1+c_2\ln x)x^{-1} + x^{-1}\left[\ln x - \ln(1-x)\right] \\ &\Longrightarrow y = (c_1+c_2\ln x)x^{-1} + x^{-1}\ln\left(\frac{x}{1-x}\right). \end{aligned}$$

This is the required solution. □

Exercise (Assignment)

(Q.1) Solve: $x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = x^5$.

Ans. $y = c_1x^{-5} + c_2x^{-1} + \dfrac{1}{60}x^5$.

(Q.2) Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$.

Ans. $y = c_1x^4 + c_2x^{-1} + \dfrac{1}{5}x^4\ln x$.

(Q.3) Solve: $x^4\dfrac{d^4y}{dx^4} + 2x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x + \ln x$.

Ans. $y = x\left[c_1 + c_2\ln x + c_3(\ln x)^2 + c_4(\ln x)^3\right] + \dfrac{x(\ln x)^4}{4!} + \ln x + 4$.

(Q.4) Solve: $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\ln x\cdot\sin(\ln x)+1}{x}$.

Ans. $y = x^2\left[c_1x^{\sqrt{3}} + c_2x^{-\sqrt{3}}\right.$
$\left.+x^{-1}\left[\ln x\left(5\sin\ln x + 6\cos\ln x\right)/61 + \left(54\sin\ln x + 382\cos\ln x\right)/3721 + 1/6\right]\right]$.