Magnetic Field and Biot-Savart Law

Concept of Magnetic Field

Magnetic field is a region around a magnet where magnetic forces act, affecting other magnets and magnetic materials. It is represented by magnetic field lines, also called magnetic flux lines, which indicate the direction and strength of the magnetic force.

Magnetic fields are produced by electric currents, either macroscopic currents in wires or microscopic currents associated with electrons in atomic orbits.

The Lorentz force acts on a charge q moving with velocity v in the presence of electric field E and magnetic field B, given by:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

where $q\vec{E}$ is the electric force and $q(\vec{v} \times \vec{B})$ is the magnetic force.

The SI unit of magnetic field is tesla (T), where 1 tesla = 10^4 gauss.

When a charge enters a magnetic field at an angle θ, the magnetic force magnitude is:

$$F_m = qvB\sin\theta$$

Oersted's Experiment

Oersted observed that a compass needle deflects when placed near a current-carrying wire, indicating that electric current produces a magnetic field around the wire. Reversing the current reverses the deflection direction.

Biot-Savart Law

The magnetic field $d\vec{B}$ due to a small current element $Id\vec{l}$ at a point is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$$

where:

- I is the current
- $d\vec{l}$ is the vector length of the small wire segment in the direction of current
- $\hat{r}$ is the unit vector from the wire segment to the point
- r is the distance from the wire segment to the point
- $\mu_0 = 4\pi\times10^{-7}$ Wb/Am is the permeability of free space

The magnitude of the magnetic field is:

$$dB = \frac{\mu_0}{4\pi} \frac{I dl \sin\theta}{r^2}$$

where θ is the angle between $d\vec{l}$ and $\vec{r}$.

Applications of Biot-Savart Law

Magnetic field at a point on the axis of a current-carrying circular loop of radius R is:

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}$$

At the center of a coil with N turns:

$$B = \frac{\mu_0 N I}{2R}$$

Magnetic field due to a current-carrying circular arc of radius r subtending angle θ at the center is:

$$B = \frac{\mu_0 \theta I}{4\pi r}$$

For a semicircular arc, the magnetic field is half that of a full circular loop.

For two perpendicular coils carrying currents I_1 and I_2, the total magnetic field at the center is:

$$B = \frac{\mu_0}{2r}\sqrt{I_1^2 + I_2^2}$$

Solved Examples

Example 1: Calculate the magnetic field at a point 0.5 m away on the y-axis due to a small current element of length 1 cm carrying 10 A current along the x-axis.

Solution:

Given: $I = 10 \text{ A}, dl = 1 \text{ cm} = 10^{-2} \text{ m}, r = 0.5 \text{ m}, \theta = 90^\circ$

Using Biot-Savart law:

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} = 10^{-7} \times \frac{10 \times 10^{-2} \times 1}{(0.5)^2} = 4 \times 10^{-8} \text{ T}$$

The magnetic field direction is along the +z-axis by the right-hand rule.

Example 2: A straight wire carrying 12 A current is bent into a semicircular arc of radius 2 cm. Find the magnetic field at the center of the arc.

Solution:

Magnetic field due to straight segments is zero as $d\vec{l} \times \vec{r} = 0$.

Magnetic field due to semicircular arc:

$$B = \frac{\mu_0 I}{4r} = \frac{4\pi \times 10^{-7} \times 12}{4 \times 2 \times 10^{-2}} = 1.9 \times 10^{-4}\ \mathrm{T}$$

Direction is into the plane of the paper.

Example 3: Calculate the magnetic field at the center of a tightly wound coil of 100 turns, radius 10 cm, carrying 1 A current.

Solution:

$N = 100, R = 0.1\ \mathrm{m}, I = 1\ \mathrm{A}$

$$B = \frac{\mu_0 N I}{2R} = \frac{4\pi \times 10^{-7} \times 100 \times 1}{2 \times 0.1} = 6.28 \times 10^{-4}\ \mathrm{T}$$

Practice Set

- **Level 1:** What is the SI unit of magnetic field? Explain the physical meaning of magnetic field lines.
- **Level 2:** Using Biot-Savart law, explain why the magnetic field at the center of a circular loop is proportional to the current and inversely proportional to the radius.

- **Level 3:** A current element of length 2 cm carries 5 A current. Calculate the magnetic field at a point 10 cm away perpendicular to the element. Use Biot-Savart law.

Answer Key

- **Level 1:** The SI unit of magnetic field is tesla (T). Magnetic field lines represent the direction and strength of magnetic force; they emerge from the north pole and enter the south pole of a magnet.
- **Level 2:** According to Biot-Savart law, magnetic field $B \propto I$ because the magnetic field strength increases with current. It is inversely proportional to the square of the distance r^2, so at the center of the loop, $B \propto 1/R$ where R is the radius.
- **Level 3:** Given $I = 5\ \mathrm{A}, dl = 2\ \mathrm{cm} = 0.02\ \mathrm{m}, r = 0.1\ \mathrm{m}, \theta = 90^\circ$

$$B = \frac{\mu_0}{4\pi} \frac{I dl \sin\theta}{r^2} = 10^{-7} \times \frac{5 \times 0.02 \times 1}{(0.1)^2} = 1 \times 10^{-7}\ \mathrm{T}$$

Ampere's Circuital Law

Statement and Explanation

Ampere's circuital law states that the line integral of the magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the total current I enclosed by the path:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

where $d\vec{l}$ is an infinitesimal segment of the closed path.

Force on Current-Carrying Conductor

The force on a conductor of length l carrying current I in a magnetic field $\vec{B}$ is:

$$\vec{F} = I\vec{l} \times \vec{B}$$

where $\vec{l}$ is a vector in the direction of current with magnitude equal to the length of the conductor.

Forces Between Two Parallel Currents

Two parallel wires carrying currents I_1 and I_2 separated by distance r exert forces on each other. The magnetic field at the second wire due to the first is:

$$B = \frac{\mu_0 I_1}{2\pi r}$$

The force per unit length on the second wire is:

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Currents in the same direction attract; opposite directions repel.

Applications to Long Straight Wire and Solenoid

Magnetic field due to a long straight wire at distance r:

$$B = \frac{\mu_0 I}{2\pi r}$$

Magnetic field inside a solenoid with n turns per unit length carrying current I:

$$B = \mu_0 n I$$

At the ends of the solenoid:

$$B_{end} = \frac{\mu_0 n I}{2}$$

Solved Examples

Example 1: Calculate the magnetic field inside a solenoid of length 0.5 m, radius 1 cm, 500 turns, carrying 5 A current.

Solution:

Number of turns per unit length:

$$n = \frac{500}{0.5} = 1000 \text{ turns/m}$$

Magnetic field inside solenoid:

$$B = \mu_0 n I = 4\pi \times 10^{-7} \times 1000 \times 5 = 6.28 \times 10^{-3}\ \mathrm{T}$$

Motion of Charged Particle in Magnetic Field

A charged particle moving perpendicular to a uniform magnetic field experiences a centripetal force:

$$F_c = \frac{mv^2}{r} = qvB$$

Radius of circular path:

$$r = \frac{mv}{qB}$$

Angular frequency (cyclotron frequency):

$$\omega_c = \frac{qB}{m}$$

Period of revolution:

$$T = \frac{2\pi}{\omega_c} = \frac{2\pi m}{qB}$$

If the velocity has a component parallel to the magnetic field, the particle moves in a helical path.

Practice Set

- **Level 1:** State Ampere's circuital law and explain the magnetic field inside a solenoid.
- **Level 2:** Derive the expression for the magnetic field due to a long straight current-carrying wire using Ampere's law.
- **Level 3:** A proton moves in a magnetic field of 0.5 T with a velocity perpendicular to the field. Calculate the radius of its circular path if its speed is 2×10^6 m/s. (Mass of proton = 1.67×10^{-27} kg, charge = 1.6×10^{-19} C)

Answer Key

- **Level 1:** Ampere's circuital law states that the line integral of magnetic field around a closed path equals μ_0 times the current enclosed. Inside a solenoid, the magnetic field is uniform and given by $B = \mu_0 n I$.
- **Level 2:** Using Ampere's law on a circular path around the wire, $\oint B dl = B(2\pi r) = \mu_0 I$, so $B = \frac{\mu_0 I}{2\pi r}$.
- **Level 3:** Radius $r = \frac{mv}{qB} = \frac{1.67 \times 10^{-27} \times 2 \times 10^6}{1.6 \times 10^{-19} \times 0.5} = 4.18 \times 10^{-2}$ m.

Torque and Galvanometer

Torque on Current Loop

A rectangular loop of current I with area A in a magnetic field B experiences a torque:

$$\tau = nBIA \sin\theta$$

where n is the number of turns and θ is the angle between the normal to the loop and the magnetic field.

Torque is maximum when the loop is parallel to the magnetic field and zero when perpendicular.

In vector form:

$$\vec{\tau} = \vec{M} \times \vec{B}$$

where $\vec{M} = nIA\hat{n}$ is the magnetic moment.

Moving Coil Galvanometer

A moving coil galvanometer measures small electric currents by the torque produced on a current-carrying coil in a magnetic field.

The deflecting torque is:

$$\tau = nBIA$$

Current sensitivity is the deflection per unit current:

$$\frac{\theta}{I} = \frac{nBA}{C}$$

Voltage sensitivity is the deflection per unit voltage:

$$\frac{\theta}{V} = \frac{nBA}{CG}$$

where C is the torsional constant and G is the galvanometer resistance.

Conversion of Galvanometer

To convert a galvanometer into an ammeter, a low resistance (shunt) is connected in parallel.

To convert it into a voltmeter, a high resistance is connected in series.

Solved Examples

Example 1: Calculate the magnetic field at the center of a coil of 100 turns, radius 10 cm, carrying 1 A current.

Solution:

$$B = \frac{\mu_0 NI}{2R} = \frac{4\pi \times 10^{-7} \times 100 \times 1}{2 \times 0.1} = 6.28 \times 10^{-4}\ \mathrm{T}$$

Example 2: A galvanometer with resistance 60 Ω is connected in a circuit with a 3 V battery and 3 Ω resistor. Calculate the current if (a) galvanometer is used as is, (b) converted to ammeter with 0.02 Ω shunt, (c) ideal ammeter with zero resistance.

Solution:

(a) Total resistance = 60 + 3 = 63 Ω, current $I = \frac{3}{63} = 0.048\ \text{A}$

(b) Resistance of galvanometer with shunt:

$$R = \frac{R_G r_s}{R_G + r_s} = \frac{60 \times 0.02}{60 + 0.02} = 0.02\Omega$$

Total resistance = 0.02 + 3 = 3.02 Ω, current $I = \frac{3}{3.02} = 0.99\ \text{A}$

(c) Ideal ammeter resistance zero, current $I = \frac{3}{3} = 1.00\ \text{A}$

Practice Set

- **Level 1:** Define torque on a current-carrying loop in a magnetic field.
- **Level 2:** Explain how a galvanometer works and how it can be converted into an ammeter.
- **Level 3:** A coil of 200 turns, area 0.01 m², carries 0.5 A current in a magnetic field of 0.2 T. Calculate the torque when the plane of the coil is at 30° to the magnetic field.

Answer Key

- **Level 1:** Torque on a current-carrying loop is the turning effect due to magnetic forces, given by $\tau = nBIA\sin\theta$.
- **Level 2:** A galvanometer measures small currents by the torque on a coil in a magnetic field. It can be converted to an ammeter by connecting a low resistance shunt in parallel to allow larger currents.
- **Level 3:** $\tau = nBIA\sin\theta = 200 \times 0.2 \times 0.5 \times 0.01 \times \sin 30^\circ = 0.1\ \text{Nm}.$

Quick Reference Table

Magnetic Field and Forces

- Magnetic field unit: Tesla (T)
- Biot-Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$
- Magnetic field due to long straight wire: $B = \frac{\mu_0 I}{2\pi r}$
- Magnetic field inside solenoid: $B = \mu_0 nI$
- Force on current-carrying conductor: $\vec{F} = I\vec{l}\times\vec{B}$
- Force between parallel currents: $\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$

Motion of Charged Particles

- Radius of circular path: $r = \frac{mv}{qB}$
- Cyclotron frequency: $\omega_c = \frac{qB}{m}$
- Period of revolution: $T = \frac{2\pi m}{qB}$

Torque and Galvanometer

- Torque on loop: $\tau = nBIA\sin\theta$
- Current sensitivity: $\frac{\theta}{I} = \frac{nBA}{C}$
- Voltage sensitivity: $\frac{\theta}{V} = \frac{nBA}{CG}$

Common Mistakes and Misconceptions

- Confusing magnetic field direction with electric field direction; magnetic field is always perpendicular to velocity of charged particle.
- Forgetting that magnetic force does no work; it changes direction but not speed of charged particle.
- Assuming magnetic field inside solenoid is zero; it is uniform and strong inside.

- Mixing up units of magnetic field; tesla and gauss differ by factor of 10^4.
- Incorrectly applying Biot-Savart law without considering vector cross product and angle.

Glossary

- **Magnetic Field:** A vector field around magnets and currents that exerts magnetic forces.
- **Biot-Savart Law:** A law that relates magnetic field to current elements.
- **Ampere's Circuital Law:** Relates magnetic field around a closed loop to current enclosed.
- **Torque:** A turning force causing rotation.
- **Galvanometer:** Instrument to detect and measure small electric currents.
- **Magnetic Moment:** A measure of the strength and orientation of a magnetic source.
- **Cyclotron Frequency:** Frequency of charged particle's circular motion in magnetic field.