Rational Numbers

A rational number is any number that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. The set of rational number

Examples include $\frac{1}{2}, \frac{3}{4}, \frac{4}{5}, -\frac{2}{3}$, etc.

Symbolically, $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$.

Decimal Expansion of Rational Numbers

The decimal expansion of a rational number is either terminating or non-terminating recurring.

If the decimal expansion terminates, it is called a terminating decimal. If it repeats periodically, it is called a non-terminating recurring d

Terminating Decimals

When dividing p by q, if the remainder becomes zero, the decimal expansion terminates.

Example: $\frac{1}{2} = 0.5$, $\frac{52}{100} = 0.52$.

A rational number $\frac{p}{q}$ is a terminating decimal if q can be expressed as $2^n \times 5^m$ where n, m are whole numbers.

Examples:

- $\frac{5}{8} = \frac{5}{2^3 \times 5^0}$ is terminating.
- $\frac{9}{1280} = \frac{9}{2^8 \times 5^1}$ is terminating.
- $\frac{4}{45} = \frac{4}{3^2 \times 5^1}$ is not terminating.

Non-Terminating Recurring Decimals

If the denominator q has prime factors other than 2 and 5, the decimal expansion is non-terminating recurring.

Examples:

- $\frac{4}{45} = 0.0\overline{8}$
- $\frac{5}{21} = 0.\overline{238095}$

$= .\overline{238095}$

Properties of Rational Numbers

- Every integer is a rational number.
- There are infinitely many rational numbers between any two rational numbers.
- If $x, y \in \mathbb{Q}$, then $x + y, x - y, xy, \frac{x}{y}$ (for $y \neq 0$) are rational.

Example 1: Express $0.\overline{5}$ in the form $\frac{p}{q}$

Let $x = 0.\overline{5} = 0.555\ldots$

Multiply both sides by 10 (since one digit repeats):

$10x = 5.555\ldots$

Subtracting, $10x - x = 5.555... - 0.555... \Rightarrow 9x = 5 \Rightarrow x = \frac{5}{9}$

Irrational Numbers

An irrational number cannot be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Examples include $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi, \sqrt{2} + \sqrt{5}$, etc.

The decimal expansion of an irrational number is non-terminating and non-recurring.

Operations involving rational and irrational numbers (addition, subtraction, multiplication, division) result in irrational numbers.

Example 2: Locate $\sqrt{17}$ on the number line

Step 1: Express 17 as sum of squares: $17 = 16 + 1 = 4^2 + 1^2$

Step 2: Draw $OA = 4$ units on the number line and $AB = 1$ unit perpendicular to it.

Step 3: By Pythagoras theorem, $OB = \sqrt{OA^2 + AB^2} = \sqrt{16 + 1} = \sqrt{17}$

Step 4: With center O and radius OB, draw an arc cutting the number line at C. Point C represents $\sqrt{17}$.

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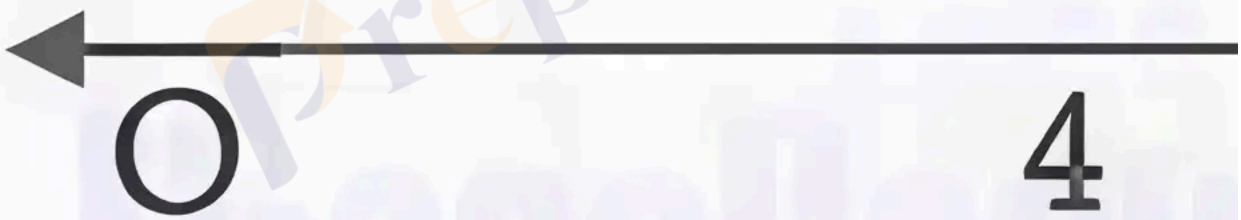
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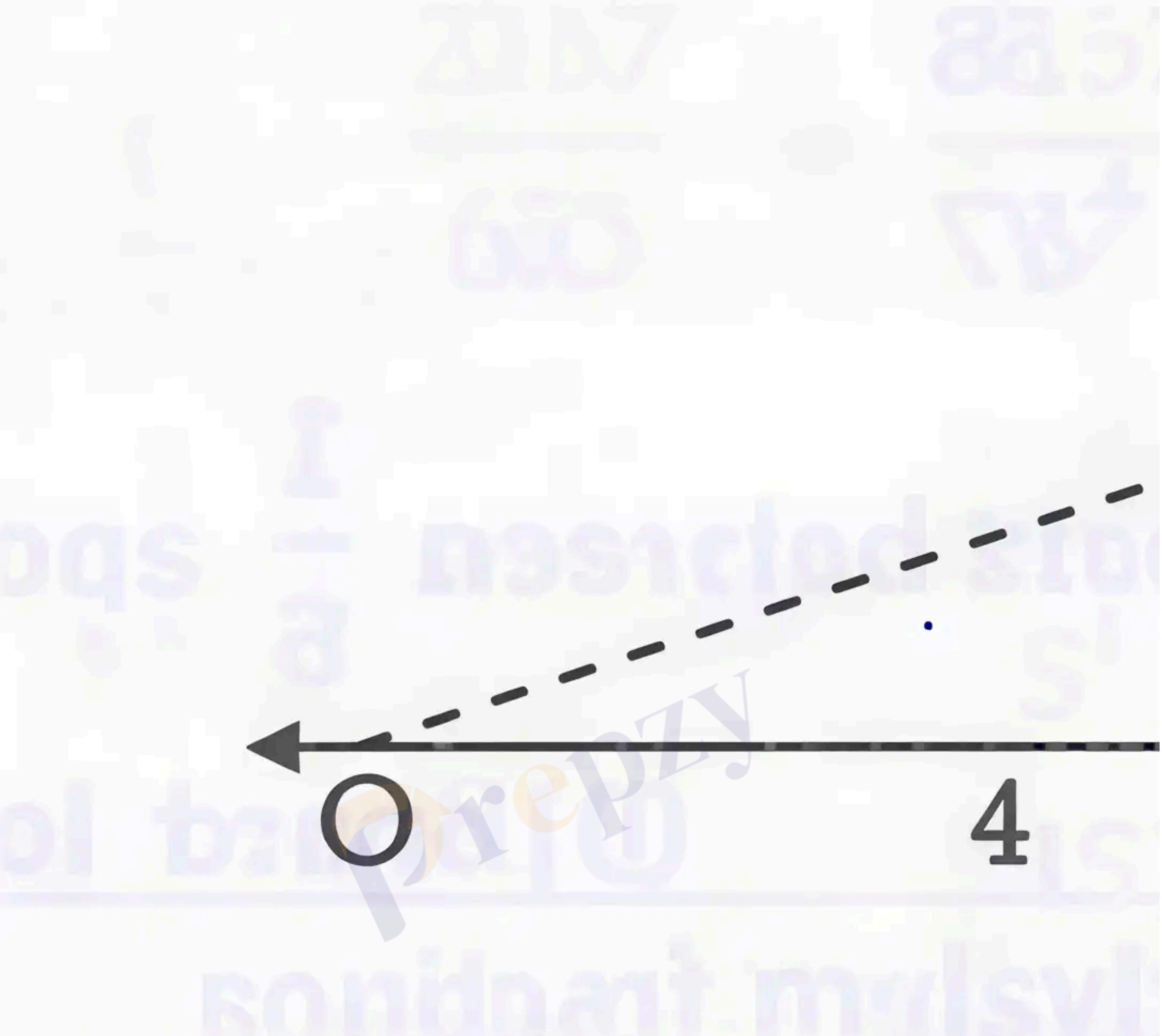

n^{th} Root of a Real Number

For real numbers a, b and positive integer n, if $a^n = b$, then a is the n^{th} root of b, denoted $\sqrt[n]{b} = a$.

Examples:

- $3^4 = 81 \Rightarrow 3 = \sqrt[4]{81}$
- $2^6 = 64 \Rightarrow 2 = \sqrt[6]{64}$

Square Root and Cube Root

Square root is the 2^{nd} root, cube root is the 3^{rd} root.

Properties:

- $\sqrt{a} \times \sqrt{a} = a$
- $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$
- $\underbrace{\sqrt[n]{a} \times \sqrt[n]{a} \times \cdots \times \sqrt[n]{a}}_{n \text{ times}} = a$

Identities for Radicals (for positive real numbers r, s)

- $\sqrt{rs} = \sqrt{r} \times \sqrt{s}$
- $\sqrt{\frac{r}{s}} = \frac{\sqrt{r}}{\sqrt{s}}$
- $(\sqrt{r} + \sqrt{s})(\sqrt{r} - \sqrt{s}) = r - s$
- $(r + \sqrt{s})(r - \sqrt{s}) = r^2 - s$

- $(\sqrt{r}-\sqrt{s})^2 = r - 2\sqrt{rs} + s$

Laws of Radicals (for positive real numbers a, b)

- $\sqrt[n]{a^n} = a$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}}$
- $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- $\sqrt[p]{\frac{a^n}{a^m}} = \sqrt[p]{a^{n-m}}$
- $\sqrt[p]{a^n} \times a^m = \sqrt[p]{a^{n+m}}$
- $\sqrt[p]{(a^n)^m} = \sqrt[p]{a^{nm}}$
- $a^{-m} = \frac{1}{a^m}$

Example 3: Simplify $\sqrt[4]{81} - 8\sqrt[3]{216} + 15\sqrt[5]{32} + \sqrt{225}$

Step 1: Express radicals as powers:

$$81^{\frac{1}{4}} - 8 \times 216^{\frac{1}{3}} + 15 \times 32^{\frac{1}{5}} + 225^{\frac{1}{2}}$$

Step 2: Factorize:

$$(3^4)^{\frac{1}{4}} - 8 \times (6^3)^{\frac{1}{3}} + 15 \times (2^5)^{\frac{1}{5}} + (15^2)^{\frac{1}{2}}$$

Step 3: Multiply exponents:

$$3^{4\times\frac{1}{4}} - 8 \times 6^{3\times\frac{1}{3}} + 15 \times 2^{5\times\frac{1}{5}} + 15^{2\times\frac{1}{2}} = 3 - 8 \times 6 + 15 \times 2 + 15$$

Step 4: Calculate:

$$3 - 48 + 30 + 15 = 0$$

Laws of Exponents with Integral Powers

For $a > 0$ and rational numbers r, s:

- $a^r \times a^s = a^{r+s}$
- $(a^r)^s = a^{rs}$
- $\frac{a^r}{a^s} = a^{r-s}$ (if $r > s$)
- $a^r b^r = (ab)^r$
- $a^{-r} = \frac{1}{a^r}$
- $a^{\frac{r}{s}} = (a^r)^{\frac{1}{s}} = (a^{\frac{1}{s}})^r$
- $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$
- $\left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^r$
- $a^0 = 1$

Examples

- $3^4 \times 3^3 = 3^7$
- $\frac{4^7}{4^2} = 4^5$
- $3^2 \times 4^2 = (3 \times 4)^2 = 12^2$
- $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2$
- $\left(\frac{1}{3}\right)^{-7} = 3^7$
- $9^{-2} = \frac{1}{9^2}$

Example 4: Find the value of $\frac{3^{40}+3^{39}+3^{38}}{3^{41}+3^{40}-3^{39}}$

Step 1: Factor out 3^{38} from numerator and 3^{39} from denominator:

$$\frac{3^{38}(3^2+3^1+1)}{3^{39}(3^2+3^1-1)} = 3^{38-39} \times \frac{9+3+1}{9+3-1} = 3^{-1} \times \frac{13}{11}$$

Step 2: Simplify:

$$= \frac{13}{3\times11} = \frac{13}{33}$$

Rationalisation of Real Numbers

Rationalisation is the process of converting a number into an equivalent form with a rational denominator.

To rationalise the denominator of $\frac{1}{\sqrt{r}+s}$, multiply numerator and denominator by the conjugate $\sqrt{r}-s$.

Example

$\frac{1}{5+\sqrt{3}} = \frac{1}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{5-\sqrt{3}}{25-3} = \frac{5-\sqrt{3}}{22}$

List of Rationalisation Factors

Term	R
$\frac{1}{\sqrt{r}}$	
$\frac{1}{\sqrt{r}-s}$	
$\frac{1}{\sqrt{r}+s}$	
$\frac{1}{\sqrt{r}-\sqrt{s}}$	
$\frac{1}{\sqrt{r}+\sqrt{s}}$	

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Example 5: Rationalise the denominator of $\frac{7}{\sqrt{5}-\sqrt{2}}$

Step 1: Let $x = \frac{7}{\sqrt{5}-\sqrt{2}}$

Step 2: The conjugate of denominator is $\sqrt{5}+\sqrt{2}$

Step 3: Multiply numerator and denominator by conjugate:

$x = \frac{7}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{7(\sqrt{5}+\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{7(\sqrt{5}+\sqrt{2})}{5-2} = \frac{7}{3}(\sqrt{5}+\sqrt{2})$