Rate of Change of Bodies

The rate of change of one variable with respect to another is a fundamental concept in calculus. If two variables x and y depend on a third variable t, such that $x = f(t)$ and $y = g(t)$, then by the Chain Rule, the rate of change of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0$$

This means the rate of change of y with respect to x can be computed using their rates of change with respect to t.

When y is a function of x, say $y = f(x)$, then $\frac{dy}{dx}$ represents the instantaneous rate of change of y with respect to x at a particular value $x = \alpha$.

Worked Illustration

Example 1: Find the rate of change of the area of a circle with respect to its radius r when $r = 5$ cm.

Solution:

The area of a circle is $A = \pi r^2$.

Differentiate A with respect to r:

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

At $r = 5$ cm,

$$\frac{dA}{dr} = 2\pi \times 5 = 10\pi$$

Thus, the area is increasing at a rate of 10π cm^2 per unit increase in radius.

Practice Set

- **Level 1 – Easy:** Find the rate of change of the circumference of a circle with respect to its radius.
- **Level 2 – Moderate:** If the radius of a sphere increases at a rate of 3 cm/s, find the rate of change of its volume when the radius is 4 cm.
- **Level 3 – Challenging:** A ladder leaning against a wall slides down at a rate of 2 m/s. Find the rate of change of the distance of the foot of the ladder from the wall when the top of the ladder is 5 m above the ground.

Answer Key

- Level 1: $\frac{dC}{dr} = 2\pi$

- Level 2: Volume $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi \times 16 \times 3 = 192\pi$ cm^3/s
- Level 3: Using Pythagoras and related rates, $\frac{dx}{dt} = \frac{5}{x} \times 2$ m/s (detailed steps required)

Quick Reference

Concept	Formula
Rate of change of y w.r.t x	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
Area of circle	$A = \pi r^2$
Rate of change of area	$\frac{dA}{dr} = 2\pi r$

Glossary

- **Rate of Change:** How one quantity changes in relation to another.
- **Chain Rule:** A rule to differentiate composite functions.
- **Instantaneous Rate:** The rate of change at a specific instant.

Increasing and Decreasing Functions

A function $f(x)$ is said to be increasing on an interval $[a, b]$ if for any $\alpha, \beta \in [a, b]$ with $\alpha > \beta$, we have $f(\alpha) > f(\beta)$. If $f'(x) \geq 0$ for all $x \in (a, b)$ and f is continuous at a and b, then f is increasing on $[a, b]$.

Similarly, $f(x)$ is decreasing on $[a, b]$ if for any $\alpha, \beta \in [a, b]$ with $\alpha > \beta$, $f(\alpha) < f(\beta)$. If $f'(x) \leq 0$ for all $x \in (a, b)$ and f is continuous at a and b, then f is decreasing on $[a, b]$.

If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

A function is called monotonic on an interval if it is either entirely increasing or entirely decreasing on that interval.

Algorithm to find intervals of increase/decrease

1. Find $f'(x)$.
2. Solve $f'(x) = 0$ to find critical points.
3. Determine the sign of $f'(x)$ in intervals defined by critical points.
4. Where $f'(x) > 0$, f is increasing; where $f'(x) < 0$, f is decreasing.

Worked Example

Example 2: Show that $f(x) = x^3 - 3x^2 + 4x$ is increasing on $\mathbb{R}$.

Solution:

Compute the derivative:

$$f'(x) = 3x^2 - 6x + 4 = 3(x^2 - 2x + \frac{4}{3})$$

Complete the square:

$$x^2 - 2x + 1 + \frac{1}{3} = (x-1)^2 + \frac{1}{3} > 0$$

Therefore, $f'(x) = 3\left((x-1)^2 + \frac{1}{3}\right) > 0$ for all x.

Hence, f is strictly increasing on $\mathbb{R}$.

Practice Set

- **Level 1 – Easy:** Determine intervals where $f(x) = 2x + 3$ is increasing or decreasing.
- **Level 2 – Moderate:** Find intervals of increase and decrease for $f(x) = x^3 - 6x^2 + 9x + 1$.
- **Level 3 – Challenging:** For $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$, find intervals where f is increasing and decreasing.

Answer Key

- Level 1: $f'(x) = 2 > 0$, so f is increasing everywhere.
- Level 2: $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. Increasing on Math input error, decreasing on $(1, 3)$.
- Level 3: $f'(x) = 4x^3 - 12x^2 + 12x - 4 = 4(x-1)^3$. Increasing on Math input error, decreasing on Math input error.

Quick Reference

Condition	Function Behavior
$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing
$f'(x) = 0$	Constant or critical point

Glossary

- **Increasing Function:** Function values rise as x increases.
- **Decreasing Function:** Function values fall as x increases.
- **Critical Point:** Point where $f'(x) = 0$ or derivative does not exist.
- **Monotonic Function:** Function that is either entirely increasing or decreasing.

Maxima and Minima

Maxima and minima are points where a function attains its highest or lowest values locally or globally.

Let $f(x)$ be defined on an interval I.

- **Maximum value:** $f(c)$ is a maximum if $f(c) \geq f(x)$ for all $x \in I$.
- **Minimum value:** $f(c)$ is a minimum if $f(c) \leq f(x)$ for all $x \in I$.
- **Extreme value:** Either a maximum or minimum value.

Local maxima/minima: Points where $f(c)$ is maximum/minimum in a neighborhood around c.

Critical points: Points where $f'(c) = 0$ or f' does not exist.

Worked Example

Example 3: Find the minimum and maximum values of $f(x) = x^2$ on $\mathbb{R}$.

Solution:

$f'(x) = 2x$. Setting $f'(x) = 0$ gives $x = 0$ as critical point.

Since $f(x) = x^2 \geq 0$ for all x, minimum value is 0 at $x = 0$.

There is no maximum value as $f(x) \to \infty$ as $x \to \pm\infty$.

First Derivative Test

1. Find $f'(x)$.

2. Find critical points by solving $f'(x) = 0$.
3. Check sign changes of $f'(x)$ around critical points:
   - If f' changes from positive to negative, local maximum.
   - If f' changes from negative to positive, local minimum.
   - If no sign change, point of inflection.

Second Derivative Test

For twice differentiable f, at critical point c:

- If $f'(c) = 0$ and $f''(c) < 0$, local maximum at c.
- If $f'(c) = 0$ and $f''(c) > 0$, local minimum at c.
- If $f''(c) = 0$, test is inconclusive; use first derivative test.

Absolute Maxima and Minima

If f is continuous on a closed interval $[a, b]$, then f attains absolute maximum and minimum values at critical points or endpoints.

Algorithm to find absolute extrema

1. Find critical points in $[a, b]$.
2. Evaluate f at critical points and endpoints.
3. Compare values to identify absolute maximum and minimum.

Worked Example

Example 4: Find local maxima and minima of $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Solution:

Compute first derivative:

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$$

Critical points at $x = 0, 1, -2$.

Compute second derivative:

$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

Evaluate at critical points:

- $f''(0) = -24 < 0$ → local maximum at $x = 0$, $f(0) = 12$.
- $f''(1) = 36 > 0$ → local minimum at $x = 1$, $f(1) = 7$.
- $f''(-2) = 72 > 0$ → local minimum at $x = -2$, $f(-2) = -20$.

Practice Set

- **Level 1 – Easy:** Find local maxima and minima of $f(x) = x^2 - 4x + 3$.
- **Level 2 – Moderate:** Find local extrema of $f(x) = x^3 - 3x + 1$.
- **Level 3 – Challenging:** Find absolute maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 4]$.

Answer Key

- Level 1: $f'(x) = 2x - 4 = 0 \Rightarrow x = 2$. $f''(2) = 2 > 0$ local minimum at $x = 2$.
- Level 2: $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(1) = 6 > 0$ local minimum, $f''(-1) = -6 < 0$ local maximum.

- Level 3: Critical points at $x = 1, 3$. Evaluate f at 0,1,3,4. Absolute max at $x = 4$, absolute min at $x = 1$.

Quick Reference

Test	Condition	Result
First Derivative	f' changes + to -	Local maximum
First Derivative	f' changes - to +	Local minimum
Second Derivative	$f'(c) = 0, f''(c) < 0$	Local maximum
Second Derivative	$f'(c) = 0, f''(c) > 0$	Local minimum

Glossary

- **Local Maximum:** Highest value in a neighborhood.
- **Local Minimum:** Lowest value in a neighborhood.
- **Absolute Maximum:** Highest value on entire domain.
- **Absolute Minimum:** Lowest value on entire domain.
- **Critical Point:** Point where derivative is zero or undefined.
- **Point of Inflexion:** Point where concavity changes and derivative does not change sign.