Basics of Differential Equations

A differential equation is an equation involving derivatives of a function with respect to one or more independent variables. The derivative of the function is denoted by $\frac{dy}{dx}$.

Order of a differential equation is the order of the highest derivative present in the equation.

Degree of a differential equation is the power of the highest order derivative after removing radicals and fractions.

Notation for derivatives:

- $\frac{dy}{dx} = y'$
- $\frac{d^2y}{dx^2} = y''$
- $\frac{d^3y}{dx^3} = y'''$
- For higher order derivatives, $y^{(n)} = \frac{d^ny}{dx^n}$

Order and degree are always positive integers.

Example 1

Find the order and degree of the differential equation $y^{(3)} + y^2 + e^{y'} = 0.$

Solution: The highest order derivative is $y^{(3)}$, so the order is 3. The equation is not a polynomial in derivatives due to the exponential term $e^{y'}$, so the degree is not defined.

Practice Set

- Find the order and degree of $\frac{d^2y}{dx^2} + 3y' + y = 0.$
- Determine the order and degree of $(y'')^2 + y = 5.$
- Find the order and degree of $\sqrt{y''} + y = 0.$

Answer Key

- Order = 2, Degree = 1
- Order = 2, Degree = 2
- Order = 2, Degree not defined (due to square root)

Quick Reference

Term	Definition
Order	Highest order derivative in the equation
Degree	Power of highest order derivative after simplification

Glossary

- **Derivative:** Rate of change of a function.
- **Order:** Highest derivative order in the equation.

- **Degree:** Power of the highest order derivative.

Variable Separable Methods

General Solution: Solution containing arbitrary constants equal to the order of the differential equation.

Particular Solution: Solution obtained by assigning specific values to arbitrary constants in the general solution.

A differential equation is separable if it can be written as $f(x)dx = g(y)dy$. The solution is obtained by integrating both sides:

$$\int f(x)dx = \int g(y)dy + k$$

where k is the constant of integration.

Example 2

Find the general solution of $\frac{dy}{dx} = x$.

Solution:

Separate variables:

$dy = xdx$

Integrate both sides:

$$\int dy = \int x dx$$

$$y = \frac{x^2}{2} + c$$

Practice Set

- Solve $\frac{dy}{dx} = 3x^2$.
- Solve $\frac{dy}{dx} = \frac{y}{x}$.
- Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$.

Answer Key

- $y = x^3 + c$
- Separate variables: $\frac{dy}{y} = \frac{dx}{x}$, integrate to get $\ln|y| = \ln|x| + c$, so $y = kx$.
- Separate variables: $y^2 dy = x^2 dx$, integrate to get $\frac{y^3}{3} = \frac{x^3}{3} + c$, so $y^3 = x^3 + k$.

Quick Reference

Step	Action
1	Rewrite as $f(x)dx = g(y)dy$
2	Integrate both sides
3	Add constant of integration

Glossary

- **General Solution:** Solution with arbitrary constants.
- **Particular Solution:** Solution with constants assigned specific values.
- **Separable Equation:** Equation that can be written as product of functions of x and y separately.

Linear Differential Equations

A first order linear differential equation in y is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only.

Solution method:

1. Find the integrating factor (I.F.):

$$I.F. = e^{\int P(x)dx}$$

2. Multiply the entire differential equation by the I.F.:

$$I.F. \cdot \frac{dy}{dx} + I.F. \cdot P(x)y = I.F. \cdot Q(x)$$

3. Left side becomes derivative of $y \cdot I.F.$:

$$\frac{d}{dx}(y \cdot I.F.) = I.F. \cdot Q(x)$$

4. Integrate both sides:

$$y \cdot I.F. = \int I.F. \cdot Q(x)dx + c$$

5. Finally, solve for y:

$$y = \frac{1}{I.F.}\left(\int I.F. \cdot Q(x)dx + c\right)$$

Similarly, a linear differential equation in x is of the form:

$$\frac{dx}{dy} + P(y)x = Q(y)$$

and solved by analogous steps using integrating factor $e^{\int P(y)dy}$.

Example 3

Solve $\frac{dy}{dx} + y = x$.

Solution:

Given $P(x) = 1, Q(x) = x$.

Integrating factor:

$$I.F. = e^{\int 1dx} = e^x$$

Multiply both sides by e^x:

$$e^x \frac{dy}{dx} + e^x y = xe^x$$

Left side is derivative of ye^x:

$$\frac{d}{dx}(ye^x) = xe^x$$

Integrate both sides:

$$ye^x = \int xe^x dx + c$$

Use integration by parts for $\int xe^x dx$:

Let $u = x$, $dv = e^x dx$, then $du = dx$, $v = e^x$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c_1$$

Therefore,

$$ye^x = xe^x - e^x + c$$

Divide both sides by e^x:

$$y = x - 1 + ce^{-x}$$

Practice Set

- Solve $\frac{dy}{dx} + 2y = e^{-x}$.
- Solve $\frac{dy}{dx} - y = x$.
- Solve $\frac{dx}{dy} + 3x = y^2$.

Answer Key

- Integrating factor e^{2x}, solution $y = e^{-2x} \int e^x dx + ce^{-2x} = \frac{1}{3}e^{-x} + ce^{-2x}$.
- Integrating factor e^{-x}, solution $y = e^x \int xe^{-x} dx + ce^x = -x - 1 + ce^x$.
- Integrating factor e^{3y}, solution $xe^{3y} = \int y^2 e^{3y} dy + c$ (integration by parts required).

Quick Reference

Step	Action
1	Write in standard form $\frac{dy}{dx} + P(x)y = Q(x)$
2	Calculate integrating factor $e^{\int P(x)dx}$
3	Multiply equation by integrating factor
4	Integrate both sides
5	Solve for y

Glossary

- **Integrating Factor:** Function used to simplify linear differential equations.
- **Linear Differential Equation:** Equation linear in the dependent variable and its derivatives.

Homogeneous Differential Equations

A first order differential equation is homogeneous if it can be written as:

$$\frac{dy}{dx} = f(x, y)$$

where $f(kx, ky) = k^n f(x, y)$ for some degree n.

Method of solution:

Substitute $y = vx$ (where v is a function of x) so that:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Rewrite the differential equation in terms of v and x, then separate variables and integrate.

Example 4

Solve $y' = \frac{x+y}{x}$.

Solution:

Rewrite:

$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

Substitute $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substitute into equation:

$$v + x\frac{dv}{dx} = 1 + v$$

Rearranged:

$$x\frac{dv}{dx} = 1$$

Separate variables:

$$\frac{dv}{dx} = \frac{1}{x} \implies dv = \frac{dx}{x}$$

Integrate both sides:

$$\int dv = \int \frac{dx}{x} \implies v = \ln|x| + c$$

Recall $v = \frac{y}{x}$, so:

$$\frac{y}{x} = \ln|x| + c \implies y = x\ln|x| + cx$$

Practice Set

- Solve $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$.
- Solve $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$.
- Solve $\frac{dx}{dy} = \frac{x+y}{y}$.

Answer Key

- Substitute $y = vx$, solve to get $y = cx$.
- Rewrite and substitute, integrate to find implicit solution involving logarithms.
- Substitute $x = vy$, solve for v and integrate accordingly.

Quick Reference

Step	Action

1	Check if $f(kx, ky) = k^n f(x, y)$
2	Substitute $y = vx$ or $x = vy$
3	Rewrite $\frac{dy}{dx}$ or $\frac{dx}{dy}$ in terms of v and x or y
4	Separate variables and integrate

Glossary

- **Homogeneous Function:** Function satisfying $f(kx, ky) = k^n f(x, y)$.
- **Substitution:** Replacing variables to simplify the equation.