Basic Algebra of Vectors

A vector is a physical quantity having both magnitude and direction, denoted as $\vec{A}, \vec{B}, \vec{a}$. The magnitude of a vector $\vec{a}$ is denoted by $|\vec{a}|$.

Initial and Terminal Points: The initial point is where the vector originates, and the terminal point is where it ends.

Position Vector: For a point $P(x, y, z)$, the position vector is $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with magnitude $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

For points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

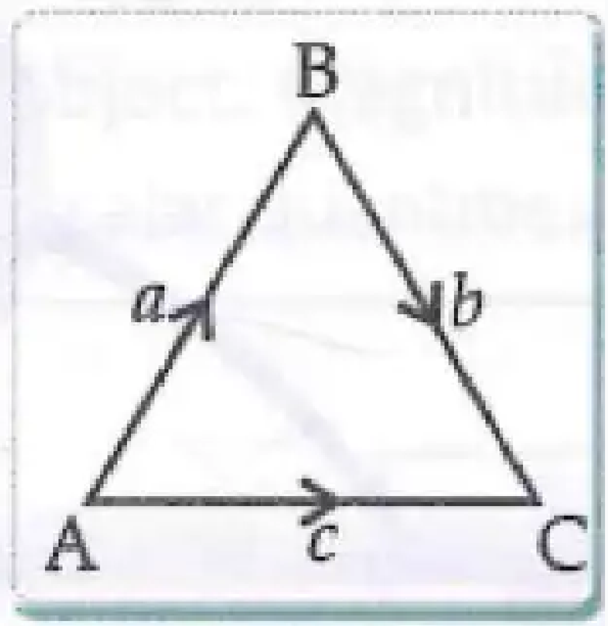
Example 1

Given $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$, check if $\vec{a} = \vec{b}$.

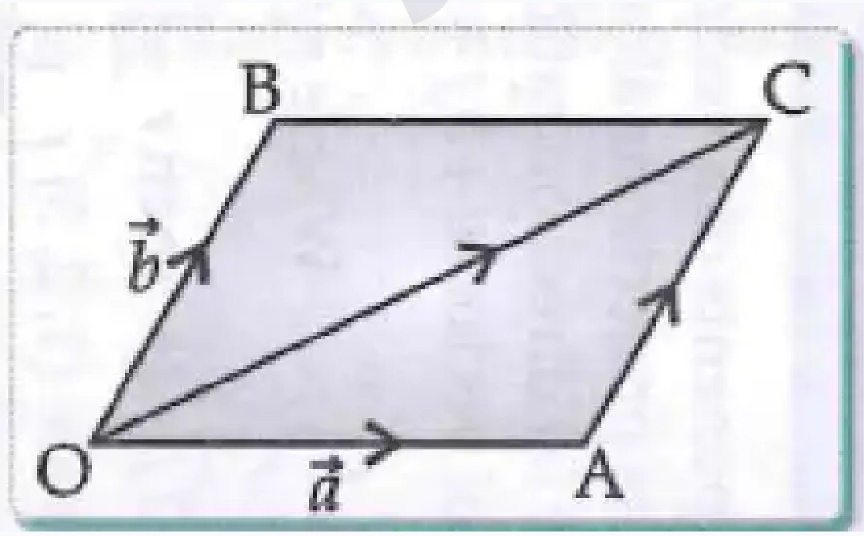
Solution: Since components differ, $\vec{a} \neq \vec{b}$.

Addition of Vectors

Triangular Law: If $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$, then $\vec{AC} = \vec{a} + \vec{b}$. Also, $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.

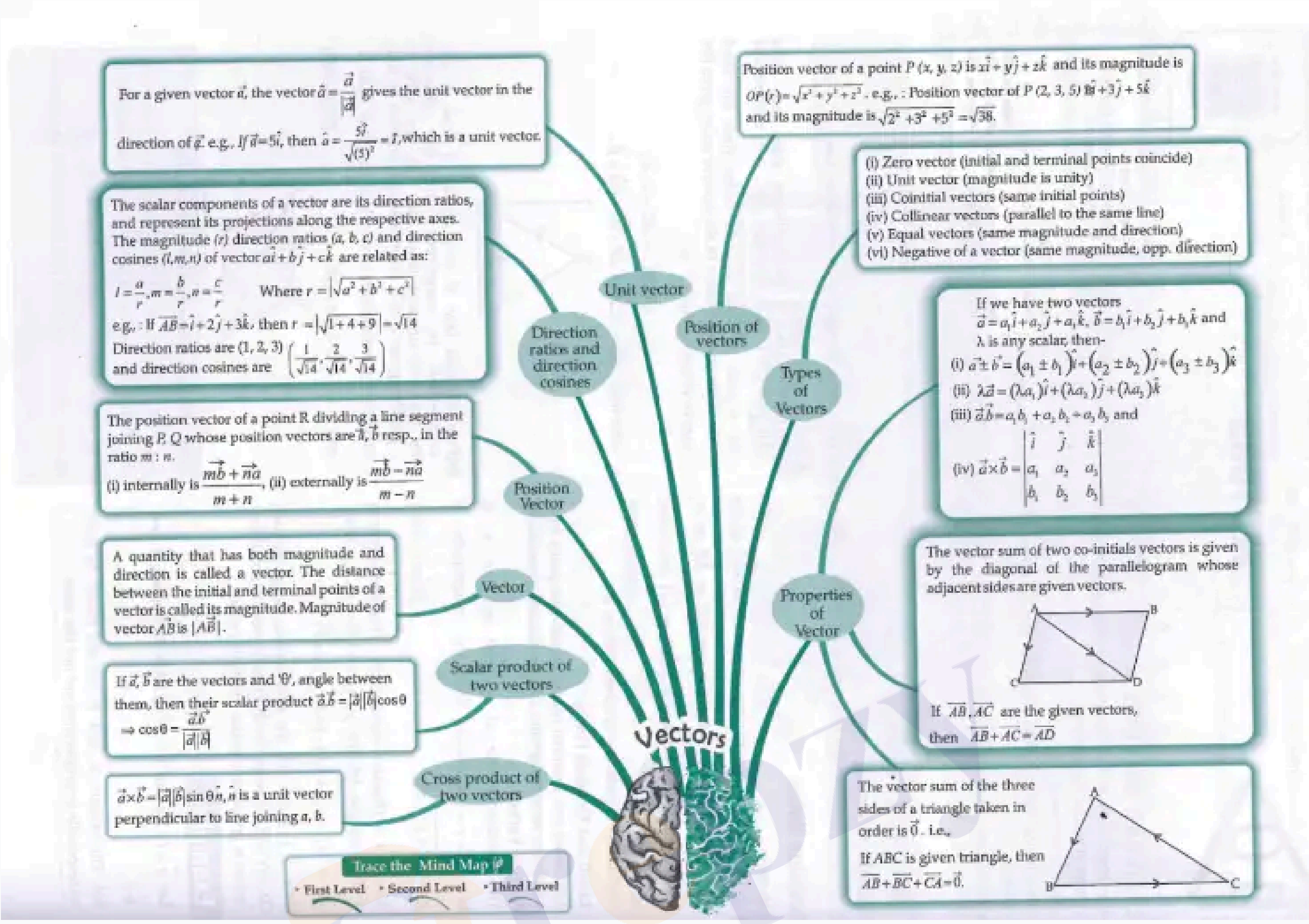


Parallelogram Law: If $\vec{a}$ and $\vec{b}$ are adjacent sides of parallelogram $OACB$, then $\vec{OC} = \vec{a} + \vec{b}$.



Multiplication of a Vector by a Scalar

For scalar k and vector $\vec{a}$, $k\vec{a}$ has magnitude $|k||\vec{a}|angle$ and direction same as $\vec{a}$ if $k > 0$, opposite if $k < 0$.



Types of Vectors

- **Zero Vector:** Initial and terminal points coincide, magnitude zero.
- **Co-initial Vectors:** Vectors with the same initial point.
- **Co-terminus Vectors:** Vectors with the same terminal point.
- **Negative Vector:** Same magnitude, opposite direction, denoted $-\vec{r}$.
- **Unit Vector:** Vector with magnitude 1, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$.
- **Reciprocal Vector:** Same direction as $\vec{r}$, magnitude $\frac{1}{|\vec{r}|}$, denoted $\vec{r}^{-1}$.
- **Equal Vectors:** Same magnitude and direction.
- **Collinear Vectors:** Parallel vectors, $\vec{a} = \lambda\vec{b}$ for scalar λ.
- **Free Vectors:** Vectors that can be moved parallel without changing magnitude or direction.

Properties of Vectors

The sum of vectors around a triangle is zero: $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.

Key Formulae

Position vector of point P dividing segment AB in ratio $m : n$:

- Internally: $\vec{OP} = \frac{m\vec{b}+n\vec{a}}{m+n}$
- Externally: $\vec{OP} = \frac{m\vec{b}-n\vec{a}}{m-n}$
- If P is midpoint, $\vec{OP} = \frac{\vec{a}+\vec{b}}{2}$

Practice Set

- **Level 1:** Find the magnitude of $\vec{a} = 3\hat{i} + 4\hat{j}$.
- **Level 2:** Given $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 4\hat{k}$, find $\vec{AB}$ where A and B have position vectors $\vec{a}$ and $\vec{b}$.
- **Level 3:** Prove that the sum of vectors around a triangle is zero.

Answer Key

- Level 1: $|\vec{a}| = \sqrt{3^2 + 4^2} = 5$
- Level 2: $\vec{AB} = \vec{b} - \vec{a} = (-1)\hat{i} - 1\hat{j} + 3\hat{k}$
- Level 3: Using triangle law, $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$ by vector addition.

Quick Reference

Concept	Formula
Position Vector	$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
Vector between points	$\vec{AB} = \vec{b} - \vec{a}$

Midpoint	$\frac{\vec{a}+\vec{b}}{2}$

Glossary

- **Vector:** Quantity with magnitude and direction.
- **Magnitude:** Length of a vector.
- **Unit Vector:** Vector of length 1.
- **Scalar:** A real number.
- **Position Vector:** Vector from origin to a point.

Dot Product of Vector

The dot product (scalar product) of vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$.

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Projection of a Vector

Projection of $\vec{a}$ on $\vec{b}$:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Projection of $\vec{b}$ on $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Example 2

Find the projection of $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution:

$$\vec{a} \cdot \vec{b} = 2 \times 1 + 3 \times 2 + 2 \times 1 = 2 + 6 + 2 = 10$$

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

Projection = $\frac{10}{\sqrt{6}} = \frac{5\sqrt{6}}{3}$

Properties of Dot Product

- $\hat{i} \cdot \hat{i} = 1, \hat{i} \cdot \hat{j} = 0$
- Dot product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

- $\vec{a} \cdot \vec{b} = 0$ if vectors are perpendicular or one is zero vector.
- Distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Key Formula

Angle between two vectors:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Practice Set

- **Level 1:** Find $\vec{a} \cdot \vec{b}$ for $\vec{a} = \hat{i} + 2\hat{j}$, $\vec{b} = 3\hat{i} + \hat{j}$.
- **Level 2:** Find the angle between $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$.
- **Level 3:** Prove that if $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are perpendicular.

Answer Key

- Level 1: $\vec{a} \cdot \vec{b} = 1 \times 3 + 2 \times 1 = 5$
- Level 2: $\cos\theta = \frac{2\times1+3\times2+1\times2}{\sqrt{4+9+1}\sqrt{1+4+4}} = \frac{2+6+2}{\sqrt{14}\sqrt{9}} = \frac{10}{3\sqrt{14}}$
- Level 3: By definition, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0$ implies $\cos\theta = 0$, so $\theta = 90°$.

Quick Reference

Concept	Formula
Dot Product	$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
Projection of $\vec{a}$ on $\vec{b}$	$\frac{\vec{a} \cdot \vec{b}}{\lvert\vec{b}\rvert}$

Angle between vectors	$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}$

Glossary

- **Dot Product:** Scalar product of two vectors.
- **Projection:** Shadow or component of one vector on another.
- **Angle between vectors:** The angle formed by two vectors.

Cross Product

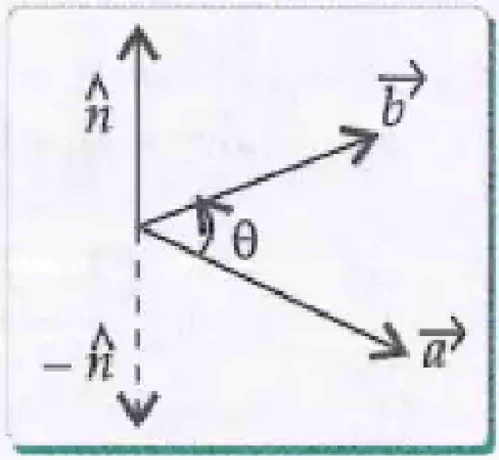
The cross product (vector product) of vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ following the right-hand rule.

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$



Properties of Cross Product

- $\hat{i} \times \hat{i} = \vec{0}$, similarly for $\hat{j}$ and $\hat{k}$.
- $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$.
- $\vec{a} \times \vec{b} = \vec{0}$ if $\vec{a}$ and $\vec{b}$ are parallel or zero vectors.
- Cross product is not commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.
- Distributive over addition: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Relationship Between Vector and Scalar Products (Lagrange's Identity)

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

Cauchy-Schwarz Inequality

For any vectors $\vec{a}$ and $\vec{b}$,

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

Applications

- Area of triangle with adjacent sides $\vec{a}$ and $\vec{b}$: $\frac{1}{2}|\vec{a} \times \vec{b}|$
- Area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$: $|\vec{a} \times \vec{b}|$

Practice Set

- **Level 1:** Compute $\hat{i} \times \hat{j}$.
- **Level 2:** Find $\vec{a} \times \vec{b}$ for $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = 4\hat{i} + 5\hat{j} + 6\hat{k}$.
- **Level 3:** Prove Lagrange's identity for vectors $\vec{a}$ and $\vec{b}$.

Answer Key

- Level 1: $\hat{i} \times \hat{j} = \hat{k}$
- Level 2: $\vec{a} \times \vec{b} = (-3)\hat{i} + 6\hat{j} - 3\hat{k}$
- Level 3: Use vector algebra to expand $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2$ and show it equals $|\vec{a}|^2|\vec{b}|^2$.

Quick Reference

Concept	Formula
Cross Product	$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
Area of Triangle	$\frac{1}{2}|\vec{a} \times \vec{b}|$
Lagrange's Identity	$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

Glossary

- **Cross Product:** Vector perpendicular to two given vectors.
- **Unit Vector:** Vector of length 1.
- **Lagrange's Identity:** Relation between dot and cross products.
- **Cauchy-Schwarz Inequality:** Bound on dot product magnitude.