Direction Ratios and Direction Cosines

Direction cosines of a line are the cosines of the angles which the line makes with the positive directions of the X, Y, and Z axes. If α, β, γ are these angles, then the direction cosines are $\cos\alpha, \cos\beta, \cos\gamma$.

These satisfy the fundamental relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Direction ratios are any three numbers a, b, c proportional to the direction cosines l, m, n respectively, i.e.,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

where l, m, n are the direction cosines.

Formula Derivation

Given direction ratios a, b, c, the magnitude $\lambda = \sqrt{a^2+b^2+c^2}$. Dividing each direction ratio by λ gives the direction cosines.

Worked Illustration

Find the direction cosines of a line with direction ratios 2, -1, -2.

Solution:

Calculate $\lambda = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$.

Therefore, direction cosines are:

$$\frac{2}{3}, \quad \frac{-1}{3}, \quad \frac{-2}{3}$$

Solved Example

Find the direction cosines of the x-axis.

Solution:

The x-axis makes angles 0°, 90°, and 90° with the x, y, and z axes respectively.

Thus, direction cosines are:

$$\cos 0^\circ = 1, \quad \cos 90^\circ = 0, \quad \cos 90^\circ = 0$$

Practice Set

- **Level 1 – Easy:** Find the direction cosines of the y-axis.
- **Level 2 – Moderate:** A line has direction ratios 3, 4, 12. Find its direction cosines.

- **Level 3 – Challenging:** Prove that the direction cosines satisfy $l^2 + m^2 + n^2 = 1$ for any line in space.

Answer Key

- Level 1: Direction cosines of y-axis are $0, 1, 0$.
- Level 2: $\lambda = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$. Direction cosines are $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.
- Level 3: Using the definition of direction cosines as cosines of angles with axes, sum of squares equals 1 by Pythagorean identity.

Quick Reference

Quantity	Formula
Direction Cosines	$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$
Relation	$l^2 + m^2 + n^2 = 1$
Direction Ratios	a, b, c proportional to l, m, n
Conversion	$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

Glossary

- **Direction Cosines:** Cosines of the angles a line makes with coordinate axes.
- **Direction Ratios:** Any three numbers proportional to the direction cosines.
- **Supplementary Angles:** Two angles whose sum is 180°.

Equations of Lines in Different Forms

Consider two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on a line. Let $\vec{a}$ and $\vec{b}$ be their position vectors, and $\vec{r}$ be the position vector of any point $P(x, y, z)$ on the line.

Vector Equation of a Line

Since points A, B, P are collinear, vectors $\vec{AP} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{b} - \vec{a}$ are collinear. Hence,

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \quad \lambda \in \mathbb{R}$$

Cartesian Equation of a Line

From the vector equation, equate components:

$$x = x_1 + \lambda(x_2 - x_1), \quad y = y_1 + \lambda(y_2 - y_1), \quad z = z_1 + \lambda(z_2 - z_1)$$

Eliminating λ, we get the Cartesian form:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Solved Example

Find the vector and Cartesian equations of the line passing through point $(5, 2, -4)$ and parallel to vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Solution:

Vector equation:

$$\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

Cartesian equations:

$$\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8} = \lambda$$

Practice Set

- **Level 1 – Easy:** Write the vector equation of a line passing through $(1, 0, 2)$ and parallel to $\hat{i} - \hat{j} + 2\hat{k}$.
- **Level 2 – Moderate:** Find the Cartesian equation of the line passing through points $(2, 3, 1)$ and $(4, 7, 5)$.
- **Level 3 – Challenging:** Find the vector and Cartesian equations of the line passing through $(1, -1, 2)$ and perpendicular to the vector $2\hat{i} + 3\hat{j} - \hat{k}$.

Answer Key

- Level 1: $\vec{r} = (1\hat{i} + 0\hat{j} + 2\hat{k}) + \lambda(\hat{i} - \hat{j} + 2\hat{k})$.
- Level 2: Cartesian equation is $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-1}{4}$.
- Level 3: Vector equation is $\vec{r} = (1\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Cartesian form as $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{-1}$.

Quick Reference

Form	Equation
Vector	$\vec{r} = \vec{a} + \lambda\vec{b}$
Cartesian	$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Glossary

- **Position Vector:** Vector from origin to a point.
- **Collinear Vectors:** Vectors lying along the same line.
- **Parameter λ:** Scalar parameter defining points on the line.

Angle Between Two Lines

Given two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2), the angle θ between them is given by:

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

Alternatively, if direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are known, then:

$$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

Vector Form

If vector equations of lines are $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right|$$

Special Cases

- Lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- Lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Practice Set

- **Level 1 – Easy:** Find the angle between lines with direction ratios (1, 0, 0) and (0, 1, 0).
- **Level 2 – Moderate:** Find the angle between lines with direction ratios (2, -1, 2) and (1, 2, -2).
- **Level 3 – Challenging:** Given vector equations $\vec{r}_1 = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r}_2 = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$, find the angle between the lines.

Answer Key

- Level 1: 90°.

- Level 2: $\cos\theta = \frac{2\times1+(-1)\times2+2\times(-2)}{\sqrt{4+1+4}\sqrt{1+4+4}} = \frac{2-2-4}{3\times3} = \frac{-4}{9}$. $\theta = \cos^{-1}\left(\frac{4}{9}\right)$.
- Level 3: Calculate dot product and magnitudes of direction vectors to find $\cos\theta$.

Quick Reference

Formula	Expression
Angle between lines (direction ratios)	$\cos\theta = \left\lvert \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \right\rvert$
Angle between lines (direction cosines)	$\cos\theta = \lvert l_1l_2 + m_1m_2 + n_1n_2 \rvert$

Glossary

- **Dot Product:** Scalar product of two vectors.
- **Direction Ratios:** Proportional numbers defining line direction.
- **Direction Cosines:** Cosines of angles with coordinate axes.

Shortest Distance Between Two Lines

For two lines in space, the shortest distance is the length of the perpendicular segment joining them.

If lines are parallel, the shortest distance is the perpendicular distance between them.

If lines are skew (non-parallel and non-intersecting), the shortest distance d is given by:

$$d = \frac{\lvert(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)\rvert}{\lvert\vec{b}_1 \times \vec{b}_2\rvert}$$

where $\vec{a}_1, \vec{a}_2$ are position vectors of points on the two lines, and $\vec{b}_1, \vec{b}_2$ are their direction vectors.

Solved Example

Find the shortest distance between the lines:

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$$

Solution:

Calculate $\vec{b}_1 \times \vec{b}_2$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = \hat{i}(-1 \times 2 - 1 \times -5) - \hat{j}(2 \times 2 - 1 \times 3) + \hat{k}(2 \times -5 - (-1) \times 3) = 3\hat{i} - \hat{j} - 7\hat{k}$$

Magnitude:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{59}$$

Vector between points on lines:

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = \hat{i} - \hat{k}$$

Dot product:

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (3\hat{i} - \hat{j} - 7\hat{k}) \cdot (\hat{i} - \hat{k}) = 3 - 0 + 7 = 10$$

Therefore, shortest distance:

$$d = \frac{|10|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Practice Set

- **Level 1 – Easy:** Find the shortest distance between two parallel lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$.
- **Level 2 – Moderate:** Find the shortest distance between lines $\vec{r} = \hat{i} + 2\hat{j} + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} + \mu(4\hat{i} + \hat{j} - 2\hat{k})$.
- **Level 3 – Challenging:** Prove that the shortest distance between skew lines is given by the formula above.

Answer Key

- Level 1: Distance is the length of vector between points on lines projected perpendicular to direction vector.
- Level 2: Use cross product and dot product as in example to find distance.
- Level 3: Use vector projection and properties of skew lines to prove formula.

Quick Reference

Quantity	Formula
Shortest distance between skew lines	$d = \frac{\lvert(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)\rvert}{\lvert\vec{b}_1 \times \vec{b}_2\rvert}$

Glossary

- **Skew Lines:** Lines that are not parallel and do not intersect.
- **Cross Product:** Vector product of two vectors.
- **Dot Product:** Scalar product of two vectors.