Fundamental Theorem of Arithmetic

The Fundamental Theorem of Arithmetic states that every composite number can be expressed as a product of prime numbers uniquely, except for the order of the factors. This is also known as the Unique Factorisation Theorem.

In other words, any integer greater than 1 is either a prime number or can be written as a unique product of prime numbers.

Formula Derivation and Explanation

Let n be a composite number. Then,

$$n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$$

where $p_1, p_2, \ldots, p_k$ are prime numbers and $a_1, a_2, \ldots, a_k$ are their respective powers.

Worked Illustration

Express 10,626 as a product of unique prime factors.

Solution:

Prime factorisation of 10,626 is:

$$10,626 = 2 \times 3 \times 7 \times 11 \times 23$$

Prime Factorisation Method to find HCF and LCM

- Find all prime factors of the given numbers.
- HCF is the product of the smallest powers of all common prime factors.
- LCM is the product of the greatest powers of all prime factors involved.

Example

Find the HCF and LCM of 6 and 20 using prime factorisation.

Solution:

Prime factorisation of 6 = $2^1 \times 3^1$

Prime factorisation of 20 = $2^2 \times 5^1$

HCF(6, 20) = $2^1 = 2$

LCM(6, 20) = $2^2 \times 3^1 \times 5^1 = 4 \times 3 \times 5 = 60$

Practice Set

- **Level 1 – Easy:** Find the prime factorisation of 84.
- **Level 2 – Moderate:** Find the HCF and LCM of 36 and 48 using prime factorisation.
- **Level 3 – Challenging:** Find the HCF and LCM of 90, 150, and 210 using prime factorisation.

Answer Key

- Prime factorisation of 84 = $2^2 \times 3 \times 7$
- HCF(36, 48) = 12, LCM(36, 48) = 144
- HCF(90, 150, 210) = 30, LCM(90, 150, 210) = 630

Quick Reference

Term	Definition
Composite Number	Integer greater than 1 with more than two factors.
Prime Number	Natural number greater than 1 with exactly two factors: 1 and itself.
HCF	Product of smallest powers of common prime factors.
LCM	Product of greatest powers of all prime factors.

Glossary

- **Prime Factorisation:** Expressing a number as a product of prime numbers.
- **HCF (Highest Common Factor):** Largest factor common to two or more numbers.
- **LCM (Least Common Multiple):** Smallest multiple common to two or more numbers.

Irrational Numbers

An irrational number is a number that cannot be expressed as a fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Examples include $\sqrt{2}$, $\sqrt{3}$
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Properties

- If p is a prime number and divides a^2, then p divides a.
- The sum or difference of a rational and an irrational number is irrational.
- The product or quotient of a non-zero rational and an irrational number is irrational.

Step-by-Step Proof: Irrationality of $\sqrt{n}$

Step 1: Assume $\sqrt{n}$ is rational.

Then, $\sqrt{n} = \frac{a}{b}$, where $a, b \in \mathbb{Z}, b \neq 0$, and $\gcd(a, b) = 1$.

Step 2: Square both sides:

$$nb^2 = a^2$$

Step 3: Since a^2 is divisible by n, a is divisible by n. Let $a = np$, where $p \in \mathbb{Z}$.

Step 4: Substitute back:

$$(np)^2 = nb^2 \Rightarrow n^2p^2 = nb^2 \Rightarrow np^2 = b^2$$

Step 5: This implies b^2 is divisible by n, so b is divisible by n.

Step 6: Both a and b are divisible by n, contradicting $\gcd(a, b) = 1$.

Step 7: Therefore, $\sqrt{n}$ is irrational.

Example: Prove that $\sqrt{2}$ is irrational.

Solution:

Assume $\sqrt{2} = \frac{p}{q}$ in lowest terms.

Squaring both sides:

$$2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$$

Since p^2 is even, p is even. Let $p = 2r$.

Substitute back:

$$2q^2 = (2r)^2 = 4r^2 \Rightarrow q^2 = 2r^2$$

Thus, q^2 is even, so q is even.

Both p and q are even, contradicting the assumption that p/q is in lowest terms.

Hence, $\sqrt{2}$ is irrational.

Practice Set

- Level 1 – Easy: Identify whether $\sqrt{9}$ is rational or irrational.
- Level 2 – Moderate: Prove that $\sqrt{3}$ is irrational.
- Level 3 – Challenging: Prove that $\sqrt{5}$ is irrational using the step-by-step method.

Answer Key

- $\sqrt{9} = 3$, which is rational.
- Proof of irrationality of $\sqrt{3}$ follows the same steps as $\sqrt{2}$ with 3 replacing 2.
- Proof of irrationality of $\sqrt{5}$ follows the same contradiction method assuming $\sqrt{5} = \frac{a}{b}$.

Quick Reference

Property	Description
Sum/Difference	Rational ± Irrational = Irrational
Product/Quotient	Non-zero Rational × Irrational = Irrational
Irrationality Proof	Assume rational, derive contradiction on co-primality.

Glossary

- Irrational Number: Number not expressible as $\frac{p}{q}$ with integers p, q.
- Co-prime Numbers: Two numbers with no common prime factors.
- Prime Number: Number greater than 1 with exactly two factors.