Relations and Functions

Relations and functions are fundamental concepts in mathematics that describe connections between elements of sets. A relation from set A to set B is a subset of the Cartesian product $A \times B$, consisting of ordered pairs (a, b) where $a \in A$ and $b \in B$. A function is a special type of relation where each element of A is related to exactly one element of B.

Concept Explanation

Given two non-empty sets A and B, the Cartesian product $A \times B$ is defined as:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

This set contains all ordered pairs where the first element is from A and the second from B. A relation R from A to B is any subset of $A \times B$.

The domain of a relation R is the set of all first elements of the ordered pairs in R, and the range is the set of all second elements:

$$\text{Domain}(R) = \{a : (a,b) \in R\} \quad \text{and} \quad \text{Range}(R) = \{b : (a,b) \in R\}$$

The codomain is the set B itself.

Types of Relations

- **Void Relation:** The empty set $\emptyset$ is a relation with no elements.
- **Universal Relation:** The entire Cartesian product $A \times B$.
- **Identity Relation:** $I_A = \{(a,a) : a \in A\}$.
- **Reflexive Relation:** Every element relates to itself: $(a,a) \in R, \forall a \in A$.
- **Symmetric Relation:** If $(a,b) \in R$, then $(b,a) \in R$.
- **Anti-Symmetric Relation:** If $(a,b) \in R$ and $(b,a) \in R$, then $a = b$.
- **Transitive Relation:** If $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.
- **Equivalence Relation:** A relation that is reflexive, symmetric, and transitive.
- **Partial Order Relation:** Reflexive, anti-symmetric, and transitive.
- **Total Order Relation:** A partial order relation where every pair of elements is comparable.

Worked Illustration

Let $A = \{1,2,3\}$ and $B = \{f,g\}$. The Cartesian product $A \times B$ is:

$$A \times B = \{(1,f),(1,g),(2,f),(2,g),(3,f),(3,g)\}$$

A relation R could be $\{(1,f),(2,g),(3,f)\}$. The domain of R is $\{1,2,3\}$, and the range is $\{f,g\}$.

Solved Example

Example: Determine if the relation $R = \{(1,1),(2,2),(3,3),(1,2)\}$ on $A = \{1,2,3\}$ is reflexive, symmetric, and transitive.

Solution:

- Reflexive: Check if $(a,a) \in R$ for all $a \in A$. Here, $(1,1),(2,2),(3,3) \in R$, so R is reflexive.
- Symmetric: For each $(a,b) \in R$, check if $(b,a) \in R$. $(1,2) \in R$ but $(2,1) \notin R$, so R is not symmetric.
- Transitive: Check if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$. $(1,2) \in R$ and $(2,2) \in R$, so $(1,2) \in R$ which is true. No other pairs violate transitivity, so R is transitive.

Practice Set

Level 1 – Easy

- Find the Cartesian product of $A = \{x,y\}$ and $B = \{1,2\}$.
- Identify the domain and range of $R = \{(a,3),(b,4),(c,3)\}$.
- Is the relation $R = \{(1,1),(2,2)\}$ on $\{1,2\}$ reflexive?

Level 2 – Moderate

- Given $R = \{(1,2),(2,3),(1,3)\}$ on $\{1,2,3\}$, check if R is transitive.
- Determine if $R = \{(a,a),(b,b),(a,b)\}$ on $\{a,b\}$ is symmetric.
- Find the number of elements in $A \times B$ if $n(A) = 4$ and $n(B) = 3$.

Level 3 – Challenging

- Prove that the identity relation on a set A is an equivalence relation.
- Show that the empty relation on A is symmetric and transitive but not reflexive.

- Given R is a partial order on A, prove that R is reflexive, anti-symmetric, and transitive.

Answer Key

- Level 1:
    - $A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2)\}$
    - Domain = $\{a, b, c\}$, Range = $\{3, 4\}$
    - Yes, R is reflexive on $\{1, 2\}$
- Level 2:
    - R is transitive because $(1, 2)$ and $(2, 3)$ imply $(1, 3)$ which is in R
    - R is not symmetric because $(a, b) \in R$ but $(b, a) \notin R$
    - Number of elements in $A \times B = 4 \times 3 = 12$
- Level 3:
    - Identity relation is reflexive, symmetric, and transitive by definition.
    - Empty relation has no elements, so symmetric and transitive hold vacuously, but not reflexive as (a, a) not in $\emptyset$.
    - By definition, partial order is reflexive, anti-symmetric, and transitive.

Quick Reference

Term	Definition
Cartesian Product	Set of all ordered pairs (a, b) with $a \in A, b \in B$
Relation	Subset of $A \times B$
Function	Relation with exactly one output for each input
Domain	Set of all first elements in relation
Range	Set of all second elements in relation
Reflexive	$(a, a) \in R$ for all a
Symmetric	If $(a, b) \in R$, then $(b, a) \in R$
Transitive	If (a, b) and (b, c) in R, then (a, c) in R

Glossary

- **Ordered Pair:** A pair (a, b) where order matters.
- **Subset:** A set all of whose elements belong to another set.
- **Reflexive Relation:** Relation where every element relates to itself.
- **Symmetric Relation:** Relation where if a relates to b, then b relates to a.
- **Transitive Relation:** Relation where if a relates to b and b relates to c, then a relates to c.
- **Equivalence Relation:** Relation that is reflexive, symmetric, and transitive.
- **Partial Order:** Relation that is reflexive, anti-symmetric, and transitive.

Functions and Their Types

A function f from set A to set B, denoted $f : A \to B$, assigns to each element $a \in A$ exactly one element $b \in B$, called the image of a, written $f(a) = b$. The set A is the domain, B is the codomain, and the set of all images $f(a)$ is the range.

Concept Explanation

Functions can be classified based on their mapping properties:

- **One-One (Injective):** Different elements in A map to different elements in B.
- **Onto (Surjective):** Every element in B is an image of some element in A.
- **One-One Onto (Bijective):** Both injective and surjective.
- **Many-One:** Multiple elements in A map to the same element in B.

Formula Derivation

Number of functions from a set A with p elements to a set B with q elements is:

$$q^p$$

since each element of A can be mapped to any of the q elements of B.

Worked Illustrations

Identity Function: $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x$. Graph is a straight line through origin with slope 1.

Constant Function: $f(x) = c$, where c is constant. Graph is a horizontal line at $y = c$.

Modulus Function: $f(x) = |x|$, defined as:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Graph is V-shaped with vertex at origin.

Signum Function: $f(x) = \frac{|x|}{x}$ for $x \neq 0$, and $f(0) = 0$. It takes values $-1, 0, 1$ depending on sign of x.

Greatest Integer Function: $f(x) = [x]$, the greatest integer less than or equal to x. Graph is a step function.

Polynomial Functions: Examples include $f(x) = x^2$ (parabola) and $f(x) = x^3$ (cubic curve).

Exponential Function: $f(x) = a^x$, where $a > 0, a \neq 1$. Domain is $\mathbb{R}$, range is $(0, \infty)$.

Natural Exponential Function: $f(x) = e^x$, where $e \approx 2.718$.

Logarithmic Function: $f(x) = \log_a x$, inverse of exponential function, with domain $(0, \infty)$ and range $\mathbb{R}$.

Solved Examples

Example 1: Find the domain and range of $f(x) = |x|$.

Solution: Domain is all real numbers $\mathbb{R}$. Range is $[0, \infty)$ since absolute value is always non-negative.

Example 2: Determine if $f(x) = x^3$ is one-one and onto from $\mathbb{R}$ to $\mathbb{R}$.

Solution: $f(x) = x^3$ is strictly increasing, so it is one-one. For any real y, $x = \sqrt[3]{y}$ satisfies $f(x) = y$, so it is onto. Hence, f is bijective.

Practice Set

Level 1 – Easy

- Find the range of $f(x) = 5$ for all real x.
- Evaluate $f(-3)$ for $f(x) = |x|$.
- Is $f(x) = x^2$ one-one on $\mathbb{R}$?

Level 2 – Moderate

- Find the domain and range of $f(x) = \log_2 x$.
- Show that $f(x) = e^x$ is one-one and onto $(0, \infty)$.
- Find the number of functions from a set with 3 elements to a set with 2 elements.

Level 3 – Challenging

- Prove that the composition of two one-one functions is one-one.
- Show that the inverse of a bijective function is also a function.
- Prove that the sum of two even functions is even.

Answer Key

- Level 1:
  - Range of $f(x) = 5$ is $\{5\}$.
  - $f(-3) = |-3| = 3$.
  - $f(x) = x^2$ is not one-one on $\mathbb{R}$ because $f(2) = f(-2) = 4$.
- Level 2:
  - Domain of $\log_2 x$ is $(0, \infty)$, range is $\mathbb{R}$.
  - $f(x) = e^x$ is one-one because it is strictly increasing; onto $(0, \infty)$ because $e^x > 0$ for all x.
  - Number of functions = $2^3 = 8$.
- Level 3:
  - Composition of one-one functions is one-one by definition.
  - Inverse of bijection is well-defined and a function.
  - Sum of even functions f and g satisfies $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$.

Quick Reference

Function Type	Definition	Domain	Range
Identity	$f(x) = x$	$\mathbb{R}$	$\mathbb{R}$
Constant	$f(x) = c$	$\mathbb{R}$	\{c\}
Modulus	$f(x) = \|x\|$	$\mathbb{R}$	$[0, \infty)$
Signum	$f(x) = \frac{\|x\|}{x}$	$\mathbb{R} \setminus \{0\}$	\{-1, 1\}
Greatest Integer	$f(x) = [x]$	$\mathbb{R}$	$\mathbb{Z}$
Exponential	$f(x) = a^x, a > 0, a \neq 1$	$\mathbb{R}$	$(0, \infty)$

Logarithmic	$f(x) = \log_a x, a > 0, a \neq 1$	$(0, \infty)$	$\mathbb{R}$

Glossary

- **Domain:** Set of inputs for a function.
- **Range:** Set of outputs of a function.
- **Codomain:** Target set of a function.
- **One-One (Injective):** Different inputs map to different outputs.
- **Onto (Surjective):** Every element of codomain is an output.
- **Bijective:** Both one-one and onto.
- **Composition:** Applying one function after another.
- **Even Function:** $f(-x) = f(x)$.
- **Odd Function:** $f(-x) = -f(x)$.