Complex Numbers

A complex number is expressed as $z = a + ib$, where a and b are real numbers, a is the real part $\mathrm{Re}(z)$, and b is the imaginary part $\mathrm{Im}(z)$. The imaginary unit i satisfies $i^2 = -1$.

Argand Plane Representation

Complex numbers can be represented as points $P(a, b)$ on the Argand plane, where the x-axis represents the real part and the y-axis the imaginary part.

Modulus and Argument

The modulus r of z is the distance from the origin to P:

$$r = \sqrt{a^2 + b^2}$$

The argument θ is the angle between the positive real axis and the line segment OP.

Algebra of Complex Numbers

For $z_1 = a + ib$ and $z_2 = c + id$:

- Addition: $z_1 + z_2 = (a + c) + i(b + d)$
- Subtraction: $z_1 - z_2 = (a - c) + i(b - d)$
- Multiplication: $z_1 z_2 = (ac - bd) + i(ad + bc)$
- Division: $\frac{z_1}{z_2} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

Powers of i

The powers of i cycle every 4 steps:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$, and so on.

Multiplicative Inverse

The inverse of a non-zero complex number $z = a + ib$ is:

$$z^{-1} = \frac{1}{z} = \frac{a - ib}{a^2 + b^2}$$

Multiplying z by z^{-1} yields 1.

Worked Example

Find the product and quotient of $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$.

Solution:

Product:

$$z_1 z_2 = (3)(1) - (4)(-2) + i[(3)(-2) + (4)(1)] = 3 + 8 + i(-6 + 4) = 11 - 2i$$

Quotient:

$$\frac{z_1}{z_2} = \frac{(3+4i)(1+2i)}{1^2 + (-2)^2} = \frac{3 + 6i + 4i + 8i^2}{1+4} = \frac{3 + 10i - 8}{5} = \frac{-5 + 10i}{5} = -1 + 2i$$

Practice Set

- **Level 1 – Easy:** Find the modulus and argument of $z = 1 + i$.
- **Level 2 – Moderate:** Compute $(2 + 3i)^2$ and express in standard form.
- **Level 3 – Challenging:** Find the multiplicative inverse of $z = 1 - i$ and verify by multiplication.

Answer Key

- Level 1: $r = \sqrt{1^2 + 1^2} = \sqrt{2}, \theta = \frac{\pi}{4}$
- Level 2: $(2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$
- Level 3: $z^{-1} = \frac{1+i}{1^2 + (-1)^2} = \frac{1+i}{2}$. Multiplying $z \times z^{-1} = (1 - i)(\frac{1+i}{2}) = \frac{1-i^2}{2} = \frac{1+1}{2} = 1$

Quick Reference

Concept	Formula
Modulus	$r = \sqrt{a^2 + b^2}$
Argument	$\theta = \tan^{-1}(\frac{b}{a})$
Multiplicative Inverse	$z^{-1} = \frac{a-ib}{a^2+b^2}$
Power of i	$i^4 = 1$ (cycle every 4)

Glossary

- **Complex Number:** Number of the form $a + ib$.
- **Imaginary Unit:** i, where $i^2 = -1$.
- **Modulus:** Distance from origin to point $P(a, b)$.
- **Argument:** Angle with positive real axis.
- **Conjugate:** $\overline{z} = a - ib$.
- **Multiplicative Inverse:** Number which when multiplied by z gives 1.

Quadratic Equations

A quadratic equation in variable x is of the form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

where a, b, c are real numbers.

Quadratic Formula

The roots of the quadratic equation are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $D = b^2 - 4ac$ determines the nature of roots.

Nature of Roots

- If $D > 0$, roots are real and unequal.
- If $D = 0$, roots are real and equal.
- If $D < 0$, roots are complex conjugates.

Sum and Product of Roots

If roots are α and β, then:

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Solving Pure Quadratic Equations

For $ax^2 + c = 0$,

$$x^2 = -\frac{c}{a} \implies x = \pm\sqrt{-\frac{c}{a}}$$

Graph of Quadratic Function

The quadratic function $f(x) = ax^2 + bx + c$ graphs as a parabola.

- If $a > 0$, parabola opens upwards with minimum vertex.
- If $a < 0$, parabola opens downwards with maximum vertex.

The vertex coordinates are:

$$x = -\frac{b}{2a}, \quad y = f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

Worked Example

Solve $2x^2 - 4x + 2 = 0$ and determine the nature of roots.

Solution:

Calculate discriminant:

$$D = (-4)^2 - 4 \times 2 \times 2 = 16 - 16 = 0$$

Since $D = 0$, roots are real and equal.

Roots:

$$x = \frac{-(-4) \pm \sqrt{0}}{2 \times 2} = \frac{4}{4} = 1$$

Practice Set

- **Level 1 – Easy:** Solve $x^2 - 5x + 6 = 0$.
- **Level 2 – Moderate:** Find roots of $3x^2 + 2x + 1 = 0$ and state their nature.
- **Level 3 – Challenging:** For $ax^2 + bx + c = 0$, prove $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ where α, β are roots.

Answer Key

- Level 1: Roots are $x = 2$ and $x = 3$, real and unequal.
- Level 2: $D = 4 - 12 = -8 < 0$, roots are complex: $x = \frac{-2 \pm i\sqrt{8}}{6} = \frac{-1}{3} \pm \frac{i\sqrt{2}}{3}$.
- Level 3: Proof:

Given roots α, β,

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$$

Also,

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \implies \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Substitute into first equation:

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

Quick Reference

Concept	Formula
Quadratic Equation	$ax^2 + bx + c = 0$
Roots	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Discriminant	$D = b^2 - 4ac$
Sum of roots	$\alpha + \beta = -\frac{b}{a}$
Product of roots	$\alpha\beta = \frac{c}{a}$
Vertex	$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

Glossary

- **Quadratic Equation:** Polynomial equation of degree 2.
- **Discriminant:** $D = b^2 - 4ac$, determines root nature.
- **Roots:** Solutions of the quadratic equation.
- **Vertex:** Turning point of the parabola.
- **Pure Quadratic Equation:** Equation of form $ax^2 + c = 0$.