Experiments Events and Sample Space

In probability, a **random experiment** is an operation or process that leads to one of several possible outcomes, which cannot be predicted with certainty beforehand. The **sample space** (denoted by S) is the set of all possible outcomes of the experiment.

An **event** is any subset of the sample space. Events can be classified as:

- **Simple event:** Contains exactly one outcome. For example, rolling a 4 on a die.
- **Compound event:** Contains more than one outcome. For example, rolling an even number $\{2, 4, 6\}$.
- **Sure event:** The entire sample space S, which always occurs.
- **Impossible event:** The empty set $\varnothing$, which cannot occur.
- **Complementary event:** The event that an event does not occur, denoted E'.
- **Mutually exclusive events:** Two events that cannot occur simultaneously, i.e., $E_1 \cap E_2 = \varnothing$.
- **Exhaustive events:** A collection of events whose union is the entire sample space S.

Formula Derivations and Rules

For any two events A and B, the probability of their union is given by the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive, then $P(A \cap B) = 0$, so:

$$P(A \cup B) = P(A) + P(B)$$

The probability of the complement of an event A is:

$$P(A') = 1 - P(A)$$

Worked Illustration

Consider rolling a fair six-sided die. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Event E: rolling an even number $\{2, 4, 6\}$.

Number of favorable outcomes = 3

Total outcomes = 6

Probability of E:

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

Solved Example

Example: Find the probability of getting a number less than 3 when a die is rolled.

Solution:

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Event $A = \{1, 2\}$

Number of favorable outcomes = 2

Total outcomes = 6

Probability:

$$P(A) = \frac{2}{6} = \frac{1}{3}$$

Practice Set

Level 1 – Easy

- Find the probability of rolling a 5 on a fair die.
- What is the probability of getting a head when a coin is tossed?

Level 2 – Moderate

- Find the probability of getting an odd number or a number greater than 4 when a die is rolled.
- In a deck of 52 cards, find the probability of drawing a heart or a king.

Level 3 – Challenging

- Two dice are rolled. Find the probability that the sum is 7 or 11.
- Find the probability of getting at least one head in three tosses of a coin.

Answer Key

Level 1

- $P(5) = \frac{1}{6}$
- $P(Head) = \frac{1}{2}$

Level 2

- Odd numbers: $\{1, 3, 5\}$, numbers greater than 4: $\{5, 6\}$

- $P(Odd \cup > 4) = P(Odd) + P(> 4) - P(Odd \cap > 4) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$
- Hearts: 13 cards, Kings: 4 cards, King of hearts counted twice
- $P(Heart \cup King) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Level 3

- Sum 7 outcomes: $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ (6 outcomes)
- Sum 11 outcomes: $(5, 6), (6, 5)$ (2 outcomes)
- Total outcomes when two dice rolled: 36
- $P(7 \cup 11) = \frac{6+2}{36} = \frac{8}{36} = \frac{2}{9}$
- Probability of no heads in 3 tosses = $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
- Probability of at least one head = $1 - \frac{1}{8} = \frac{7}{8}$

Quick Reference

Term	Definition
Sample Space (S)	Set of all possible outcomes
Event (E)	Subset of sample space
Simple Event	Event with one outcome
Compound Event	Event with multiple outcomes
Mutually Exclusive	Events that cannot occur together
Exhaustive Events	Events covering entire sample space
Complementary Event	Event not occurring
Probability of A or B	$P(A) + P(B) - P(A \cap B)$
Probability of Complement	$1 - P(A)$

Glossary

- **Random Experiment:** An experiment with uncertain outcomes.
- **Sample Space:** The set of all possible outcomes.
- **Event:** A subset of the sample space.
- **Mutually Exclusive Events:** Events that cannot happen at the same time.
- **Exhaustive Events:** Events that cover all possible outcomes.
- **Complementary Event:** The event that an event does not occur.

Axiomatic Approach to Probability

The **axiomatic approach** defines probability as a function P from the set of all events (subsets of sample space S) to the real interval $[0, 1]$ satisfying the following axioms:

1. **Non-negativity:** For any event E, $P(E) \geq 0$.
2. **Normalization:** $P(S) = 1$.
3. **Additivity:** For any two mutually exclusive events E and F, $P(E \cup F) = P(E) + P(F)$.

Derivations

Let the sample space $S = \{E_1, E_2, \ldots, E_n\}$ where E_i are mutually exclusive elementary events.

From the axioms:

- $0 \leq P(E_i) \leq 1$ for all i.
- $\sum_{i=1}^{n} P(E_i) = 1$.
- For any event A which is a union of some E_i, $P(A) = \sum_{E_i \subseteq A} P(E_i)$.

Probability of Equally Likely Outcomes

If all outcomes are equally likely, then for each elementary event s_i,

$$P(s_i) = p, \quad \text{for all } i$$

Since $\sum_{i=1}^{n} P(s_i) = 1$,

$$p \times n = 1 \implies p = \frac{1}{n}$$

Thus, the probability of each elementary event is $\frac{1}{n}$.

Addition Rule of Probability

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For three events A, B, C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Probability of Complementary Event

For an event E, its complement $\bar{E}$ satisfies:

$$P(\bar{E}) = 1 - P(E)$$

Worked Example

Example: In a random experiment with sample space $S = \{s_1, s_2, s_3, s_4\}$, the probabilities are $P(s_1) = 0.1, P(s_2) = 0.3, P(s_3) = 0.4$. Find $P(s_4)$.

Solution:

Using normalization axiom:

$$P(s_1) + P(s_2) + P(s_3) + P(s_4) = 1$$

$$0.1 + 0.3 + 0.4 + P(s_4) = 1$$

$$P(s_4) = 1 - 0.8 = 0.2$$

Practice Set

Level 1 – Easy

- State the three axioms of probability.
- Find the probability of an event E if $P(E) = 0.7$. Find $P(E')$.

Level 2 – Moderate

- Given $P(A) = 0.4, P(B) = 0.5, P(A \cap B) = 0.2$, find $P(A \cup B)$.
- In a sample space of 5 equally likely outcomes, find the probability of an event containing 2 outcomes.

Level 3 – Challenging

- Prove that $P(A - B) = P(A) - P(A \cap B)$.
- If A, B, C are mutually exclusive events with $P(A) = 0.2, P(B) = 0.3, P(C) = 0.4$, find $P(A \cup B \cup C)$.

Answer Key

Level 1

- Axioms: Non-negativity, Normalization, Additivity.
- $P(E') = 1 - 0.7 = 0.3$

Level 2

- $P(A \cup B) = 0.4 + 0.5 - 0.2 = 0.7$
- Probability = $\frac{2}{5} = 0.4$

Level 3

- Proof: $A - B = A \cap B'$, so $P(A - B) = P(A) - P(A \cap B)$.
- Since mutually exclusive, $P(A \cup B \cup C) = 0.2 + 0.3 + 0.4 = 0.9$

Quick Reference

Property	Expression
Non-negativity	$P(E) \geq 0$
Normalization	$P(S) = 1$
Additivity	$P(E \cup F) = P(E) + P(F)$ if $E \cap F = \varnothing$
Complement	$P(E') = 1 - P(E)$
Equally likely outcomes	$P(s_i) = \frac{1}{n}$

Glossary

- **Axiom:** A fundamental principle accepted without proof.
- **Sample Space:** Set of all possible outcomes.
- **Mutually Exclusive Events:** Events that cannot occur simultaneously.
- **Normalization:** Total probability of sample space is 1.
- **Additivity:** Probability of union of mutually exclusive events is sum of their probabilities.
- **Complementary Event:** Event that an event does not occur.