- Quadratic Equations

Quadratic Equations

A quadratic equation is a polynomial equation of degree 2 in the variable x, expressed in the standard form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

where a, b, and c are real numbers. The solutions or roots of this equation are the values of x that satisfy the equation.

Formula Derivation: Quadratic Formula

To find the roots of $ax^2 + bx + c = 0$, we complete the square as follows:

Start with:

$$ax^2 + bx + c = 0$$

Divide both sides by a (since $a \neq 0$):

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Rearranged:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add $\left(\frac{b}{2a}\right)^2$ to both sides to complete the square:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Left side becomes a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square roots:

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Therefore,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Worked Illustrations and Solved Examples

Example 1: Check if $(x - 2)^2 + 1 = 2x - 3$ is a quadratic equation.

Solution:

Expand and simplify:

$$(x - 2)^2 + 1 = 2x - 3$$

$$x^2 - 4x + 4 + 1 = 2x - 3$$

$$x^2 - 4x + 5 = 2x - 3$$

Bring all terms to one side:

$$x^2 - 4x + 5 - 2x + 3 = 0$$

$$x^2 - 6x + 8 = 0$$

This is in the form $ax^2 + bx + c = 0$ with $a = 1, b = -6, c = 8$, so it is a quadratic equation.

Example 2: Find roots of $2x^2 - 5x + 3 = 0$ by factorisation.

Solution:

Multiply a and c: $2 \times 3 = 6$.

Find two numbers whose product is 6 and sum is -5: -2 and -3.

Rewrite middle term:

$$2x^2 - 2x - 3x + 3 = 0$$

Group terms:

$$2x(x-1) - 3(x-1) = 0$$

Factor:

$$(2x-3)(x-1) = 0$$

Set each factor to zero:

$$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

$$x - 1 = 0 \Rightarrow x = 1$$

Roots are 1 and $\frac{3}{2}$.

Example 3: Find roots of $6x^2 - x - 2 = 0$ using quadratic formula.

Solution:

Identify coefficients: $a = 6, b = -1, c = -2$.

Calculate discriminant:

$$D = b^2 - 4ac = (-1)^2 - 4(6)(-2) = 1 + 48 = 49$$

Apply quadratic formula:

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 7}{12}$$

Roots:

$$x = \frac{1+7}{12} = \frac{8}{12} = \frac{2}{3}$$

$$x = \frac{1-7}{12} = \frac{-6}{12} = -\frac{1}{2}$$

Practice Set

- **Level 1 – Easy**
  - Check if $x^2 + 5x + 6 = 0$ is a quadratic equation.
  - Find roots of $x^2 - 7x + 10 = 0$ by factorisation.
  - Find roots of $x^2 + 4x + 4 = 0$ using quadratic formula.
- **Level 2 – Moderate**
  - Find roots of $3x^2 - 2x - 1 = 0$ using quadratic formula.
  - Determine the nature of roots of $2x^2 + 3x + 5 = 0$ using discriminant.
  - Find roots of $4x^2 - 12x + 9 = 0$ by factorisation.
- **Level 3 – Challenging**
  - Find the roots of $5x^2 + 6x + 1 = 0$ using quadratic formula and verify by factorisation.
  - For the quadratic equation $ax^2 + bx + c = 0$, prove that sum of roots is $-\frac{b}{a}$ and product of roots is $\frac{c}{a}$.
  - Apply quadratic equations to solve: The area of a rectangle is 48 sq. units and its length is 3 units more than its width. Find the dimensions.

Answer Key

- **Level 1**
  - Yes, it is quadratic with $a = 1, b = 5, c = 6$.
  - Roots: $x = -2, -3$.
  - Roots: $x = -2, -2$ (repeated root).
- **Level 2**
  - Roots: $x = 1, -\frac{1}{3}$.
  - Discriminant $D = 3^2 - 4(2)(5) = 9 - 40 = -31 < 0$, no real roots.
  - Roots: $x = \frac{3}{2}, \frac{3}{2}$ (repeated root).
- **Level 3**
  - Quadratic formula roots: $x = -\frac{1}{5}, -1$. Factorisation: $(5x+1)(x+1) = 0$ roots match.
  - Proof:
    - Sum of roots $\alpha + \beta = -\frac{b}{a}$ from quadratic formula.
    - Product of roots $\alpha\beta = \frac{c}{a}$.
  - Let width = x, length = $x + 3$.

    Area equation: $x(x+3) = 48$

$x^2 + 3x - 48 = 0$

Using quadratic formula:

$$x = \frac{-3 \pm \sqrt{9 + 192}}{2} = \frac{-3 \pm 15}{2}$$

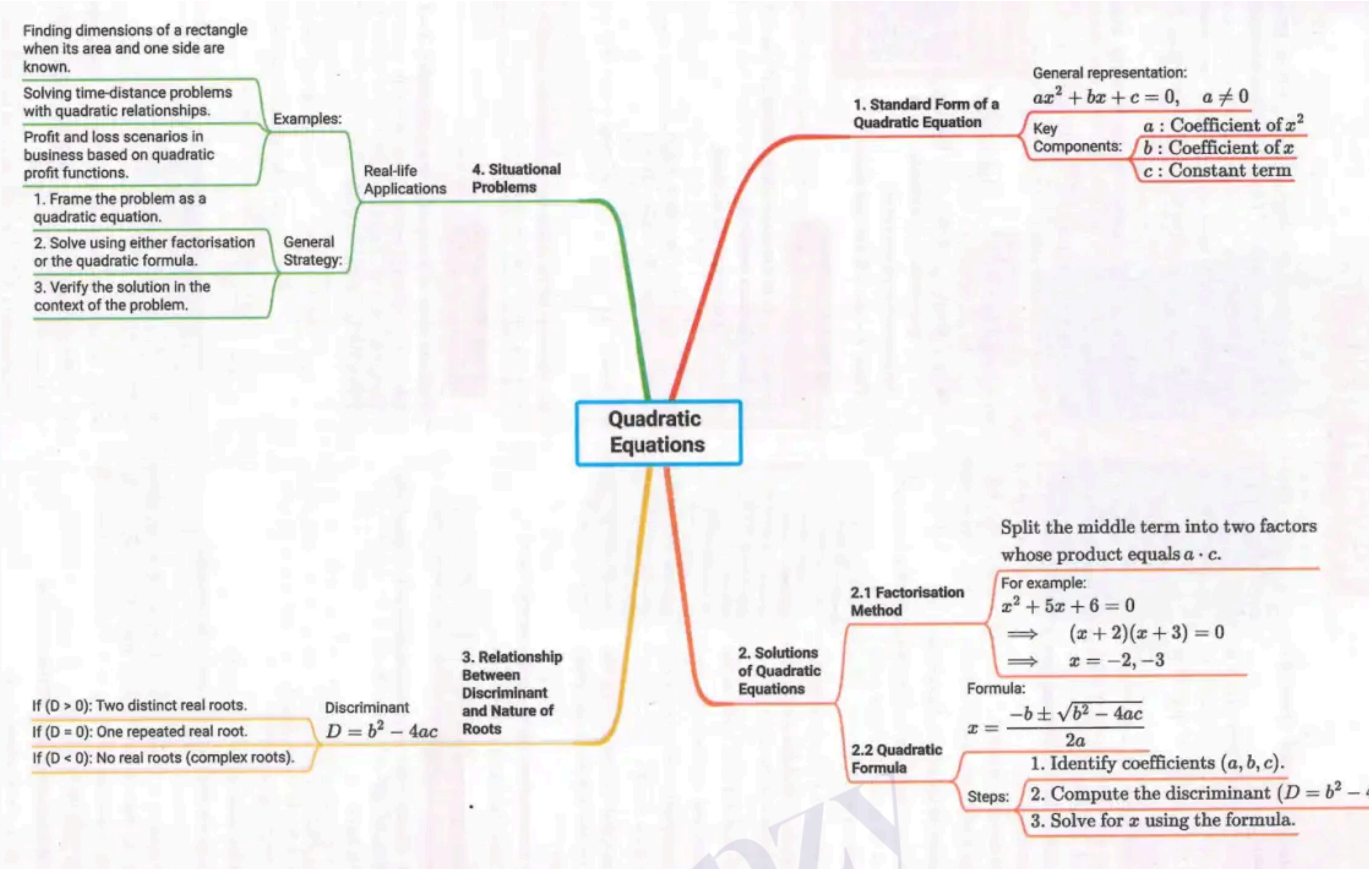
Positive root: $x = 6$, length = 9.

Quick Reference

Concept	Formula/Result
Standard form	$ax^2 + bx + c = 0, a \neq 0$
Quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Discriminant	$D = b^2 - 4ac$
Nature of roots	$D > 0$: two distinct real roots $D = 0$: two equal real roots $D < 0$: no real roots
Sum of roots	$-\frac{b}{a}$
Product of roots	$\frac{c}{a}$

Glossary

- **Quadratic Equation:** Polynomial equation of degree 2.
- **Roots/Solutions:** Values of x satisfying the quadratic equation.
- **Discriminant:** Expression $b^2 - 4ac$ determining nature of roots.
- **Factorisation:** Expressing quadratic as product of two linear factors.
- **Quadratic Formula:** Formula to find roots of quadratic equations.
- **Sum of Roots:** Sum of the solutions of the quadratic equation.
- **Product of Roots:** Product of the solutions of the quadratic equation.


Quadratic Equations
1. Standard Form of a Quadratic Equation
General representation:
$ax^2 + bx + c = 0, \quad a \neq 0$
Key Components:
a : Coefficient of x^2
b : Coefficient of x
c : Constant term
2. Solutions of Quadratic Equations
2.1 Factorisation Method
Split the middle term into two factors whose product equals $a \cdot c$.
For example:
$x^2 + 5x + 6 = 0$
$\implies (x+2)(x+3) = 0$
$\implies x = -2, -3$
2.2 Quadratic Formula
Formula:
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Steps:
1. Identify coefficients (a, b, c).
2. Compute the discriminant $(D = b^2 -$
3. Solve for x using the formula.
3. Relationship Between Discriminant and Nature of Roots
Discriminant
$D = b^2 - 4ac$
If (D > 0): Two distinct real roots.
If (D = 0): One repeated real root.
If (D < 0): No real roots (complex roots).
4. Situational Problems
Real-life Applications
Examples:
Finding dimensions of a rectangle when its area and one side are known.
Solving time-distance problems with quadratic relationships.
Profit and loss scenarios in business based on quadratic profit functions.
General Strategy:
1. Frame the problem as a quadratic equation.
2. Solve using either factorisation or the quadratic formula.
3. Verify the solution in the context of the problem.