Heron's Formula

Heron's formula is a method to calculate the area of a triangle when the lengths of all three sides are known. It applies to all types of triangles: scalene, isosceles, and equilateral.

Concept Explanation

Consider a triangle with sides of lengths a, b, and c. The perimeter P of the triangle is given by:

$$P = a + b + c$$

The semi-perimeter s is half the perimeter:

$$s = \frac{a + b + c}{2}$$

Heron's formula states that the area A of the triangle is:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Formula Derivation

Heron's formula can be derived using the law of cosines and algebraic manipulation, but here we focus on its application.

Worked Illustrations

Given a triangle with sides $a = 24$ cm, $b = 10$ cm, and perimeter $P = 60$ cm, find the area.

Step 1: Find the third side c:

$$c = P - (a + b) = 60 - (24 + 10) = 26 \text{ cm}$$

Step 2: Calculate the semi-perimeter s:

$$s = \frac{24 + 10 + 26}{2} = 30 \text{ cm}$$

Step 3: Calculate the area using Heron's formula:

$$A = \sqrt{30(30 - 24)(30 - 10)(30 - 26)} = \sqrt{30 \times 6 \times 20 \times 4}$$

$$= \sqrt{14400} = 120 \text{ cm}^2$$

Solved Examples

Example 1

Find the area of a triangle with sides 12 cm, 16 cm, and 20 cm.

Solution:

Step 1: Calculate the semi-perimeter s:

$$s = \frac{12 + 16 + 20}{2} = 24 \text{ cm}$$

Step 2: Calculate the area:

$$A = \sqrt{24(24 - 12)(24 - 16)(24 - 20)} = \sqrt{24 \times 12 \times 8 \times 4}$$

$$= \sqrt{9216} = 96 \text{ cm}^2$$

Practice Set

Level 1 – Easy

- Find the area of a triangle with sides 7 cm, 8 cm, and 9 cm.
- Calculate the area of an equilateral triangle with side length 10 cm.

Level 2 – Moderate

- A triangle has sides 15 cm, 20 cm, and 25 cm. Find its area.
- Find the area of a triangle with sides 13 cm, 14 cm, and 15 cm.

Level 3 – Challenging

- Prove that Heron's formula gives the same area as $\frac{1}{2} \times \text{base} \times \text{height}$ for a right-angled triangle with sides 6 cm, 8 cm, and 10 cm.
- Given a triangle with sides 9 cm, 40 cm, and 41 cm, find its area and verify if it is a right triangle.

Answer Key

Level 1

- $s = \frac{7+8+9}{2} = 12$, area $= \sqrt{12(12-7)(12-8)(12-9)} = \sqrt{12 \times 5 \times 4 \times 3} = \sqrt{720} \approx 26.83 \text{ cm}^2$
- For equilateral triangle, $s = \frac{10+10+10}{2} = 15$, area $= \sqrt{15(15-10)^3} = \sqrt{15 \times 5 \times 5 \times 5} = \sqrt{1875} \approx 43.30 \text{ cm}^2$

Level 2

- $s = 30$, area $= \sqrt{30(30-15)(30-20)(30-25)} = \sqrt{30 \times 15 \times 10 \times 5} = \sqrt{22500} = 150 \text{ cm}^2$
- $s = 21$, area $= \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7056} = 84 \text{ cm}^2$

Level 3

- For right triangle with sides 6, 8, 10, area by Heron's formula: $s = 12$, area $= \sqrt{12(12-6)(12-8)(12-10)} = \sqrt{12 \times 6 \times 4 \times 2} = \sqrt{576} = 24 \text{ cm}^2$. Using base and height: $\frac{1}{2} \times 6 \times 8 = 24 \text{ cm}^2$. Both match.
- For sides 9, 40, 41, $s = 45$, area $= \sqrt{45(45-9)(45-40)(45-41)} = \sqrt{45 \times 36 \times 5 \times 4} = \sqrt{32400} = 180 \text{ cm}^2$. Since $9^2 + 40^2 = 81 + 1600 = 1681$ and $41^2 = 1681$, it is a right triangle.

Quick Reference

Quantity	Formula
Semi-perimeter s	$\frac{a+b+c}{2}$
Area A	$\sqrt{s(s-a)(s-b)(s-c)}$

Glossary

- **Area (A):** The measure of the region enclosed by the triangle.
- **Sides (a, b, c):** The lengths of the three edges of the triangle.
- **Semi-perimeter (s):** Half of the triangle's perimeter.
- **Square Root ($\sqrt{}$):** A value that, when multiplied by itself, gives the original number.