Divisibility Properties and Factors

Understanding divisibility is fundamental in number theory. If a number a is divisible by another number b, then a is also divisible by all factors of b. This property helps simplify divisibility checks and factorization.

Formula Derivation

Let a, b, c be integers such that $b = c \times d$ for some integer d. If a is divisible by b, then $a = b \times k = c \times d \times k$ for some integer k. Hence, a is divisible by c.

Worked Illustration

Check if 60 is divisible by 6 and its factors.

Since 60 ÷ 6 = 10 (an integer), 60 is divisible by 6. Factors of 6 are 1, 2, 3, and 6. Check divisibility:

- 60 ÷ 2 = 30 (integer)
- 60 ÷ 3 = 20 (integer)
- 60 ÷ 1 = 60 (integer)

Thus, 60 is divisible by all factors of 6.

Solved Example

Is 84 divisible by 12 and its factors?

Step 1: Check 84 ÷ 12 = 7 (integer), so divisible by 12.

Step 2: Factors of 12 are 1, 2, 3, 4, 6, 12.

Check divisibility:

- 84 ÷ 2 = 42 (integer)
- 84 ÷ 3 = 28 (integer)
- 84 ÷ 4 = 21 (integer)
- 84 ÷ 6 = 14 (integer)
- 84 ÷ 1 = 84 (integer)

All are integers, so 84 is divisible by all factors of 12.

Practice Set

- Level 1: Check if 48 is divisible by 8 and its factors.
- Level 2: Verify if 90 is divisible by 15 and all its factors.
- Level 3: Prove that if a number is divisible by 36, it is divisible by 9 and 4.

Answer Key

- 48 ÷ 8 = 6; factors of 8 are 1, 2, 4, 8; 48 divisible by all.
- 90 ÷ 15 = 6; factors 1, 3, 5, 15; 90 divisible by all.
- If n divisible by 36, $n = 36k = 9 \times 4k$, so divisible by 9 and 4.

Quick Reference

Divisible by	Factors

6	1, 2, 3, 6
12	1, 2, 3, 4, 6, 12
15	1, 3, 5, 15

Glossary

- **Divisible:** A number a is divisible by b if $a \div b$ is an integer.
- **Factor:** A number that divides another number exactly.

Digital Roots and Divisibility Rules

Digital roots simplify divisibility checks by repeatedly summing digits until a single digit remains. Multiples of 9 always have a digital root of 9. Divisibility rules for 3, 9, and 11 use sums or differences of digits.

Formula Derivation

For a number $N = d_1 d_2 d_3 \ldots d_n$, the sum of digits is $S = \sum_{i=1}^{n} d_i$. The digital root is obtained by repeatedly summing digits of S until one digit remains.

Divisibility by 9: N divisible by 9 if S divisible by 9.

Divisibility by 3: N divisible by 3 if S divisible by 3.

Divisibility by 11: N divisible by 11 if difference between sum of digits in odd positions and even positions is divisible by 11.

Worked Illustration

Check if 729 is divisible by 9.

Sum digits: 7 + 2 + 9 = 18

Sum digits of 18: 1 + 8 = 9

Since digital root is 9, 729 is divisible by 9.

Solved Example

Is 352 divisible by 11?

Sum of digits in odd positions: 3 + 2 = 5

Sum of digits in even positions: 5

Difference: 5 - 5 = 0, which is divisible by 11.

Therefore, 352 is divisible by 11.

Practice Set

- Level 1: Find the digital root of 12345.
- Level 2: Check if 123456 is divisible by 3 and 9.
- Level 3: Verify if 2728 is divisible by 11 using the divisibility rule.

Answer Key

- Digital root of 12345: 1+2+3+4+5=15; 1+5=6
- 123456 sum digits = 1+2+3+4+5+6=21; divisible by 3 but not 9.
- 2728: odd positions sum = 2+2=4; even positions sum = 7+8=15; difference = 11; divisible by 11.

Quick Reference

Divisibility Rule	Condition
3	Sum of digits divisible by 3
9	Sum of digits divisible by 9
11	Difference of sums of digits in odd and even positions divisible by 11

Glossary

- **Digital Root:** Single digit obtained by iterative sum of digits.
- **Divisibility Rule:** Shortcut to check if a number is divisible by another without division.

Sums of Consecutive Numbers and Parity

Natural numbers can be expressed as sums of consecutive numbers. Adding or subtracting consecutive numbers results in numbers with specific parity (even or odd).

Formula Derivation

Sum of k consecutive numbers starting from n:

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$$S = n + (n+1) + (n+2) + \cdots + (n+k-1) = k \times \frac{2n+k-1}{2}$$

Parity rules:

- Sum of two consecutive numbers is always odd.
- Sum of four consecutive numbers with alternating signs (e.g., $a + b - c + d$) can be even.

Worked Illustration

Sum of 3 consecutive numbers starting from 4:

$4 + 5 + 6 = 15$

Using formula:

$k = 3, n = 4$

$$S = 3 \times \frac{2 \times 4 + 3 - 1}{2} = 3 \times \frac{8 + 2}{2} = 3 \times 5 = 15$$

Solved Example

Show that the sum of four consecutive numbers with signs $+ + - +$ is always even.

Let the numbers be $n, n+1, n+2, n+3$.

Sum = $n + (n+1) - (n+2) + (n+3) = n + n + 1 - n - 2 + n + 3 = 2n + 2$

Since $2n + 2 = 2(n+1)$, the sum is always even.

Practice Set

- Level 1: Find the sum of 5 consecutive numbers starting from 1.
- Level 2: Prove that the sum of any two consecutive numbers is odd.
- Level 3: Verify if the sum $a + b - c + d$ of four consecutive numbers is always even.

Answer Key

- Sum = 5 $\times$ $\frac{2\times1+5-1}{2}$ = 5 $\times$ $\frac{2+4}{2}$ = 5 $\times$ 3 = 15
- Two consecutive numbers: n and $n+1$, sum = $2n+1$ (odd)
- Sum $a + b - c + d = 2n + 2$ (even), as shown above.

Quick Reference

Expression	Result
Sum of k consecutive numbers	$k \times \frac{2n+k-1}{2}$
Sum of two consecutive numbers	Odd number
Sum $a + b - c + d$ of four consecutive numbers	Even number

Glossary

- **Consecutive Numbers:** Numbers that follow each other in order.
- **Parity:** Whether a number is even or odd.