Doubling a Square

To construct a square with double the area of a given square, simply doubling the side length is incorrect because the area scales with the square of the side length. Instead, the diagonal of the original square serves as the side of the new square with double the area.

Formula Derivation

Let the side length of the original square be a. Its area is a^2. The diagonal length d is given by the Pythagorean theorem:

$$d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$$

The new square constructed on this diagonal has side length $d = a\sqrt{2}$, so its area is:

$$\text{Area} = d^2 = (a\sqrt{2})^2 = 2a^2$$

This is exactly double the area of the original square.

Worked Illustration

Given a square of side 1 unit, its diagonal is $\sqrt{2}$ units. Constructing a square on this diagonal yields an area of 2 square units, double the original.

Solved Example

Given a square of side 5 cm, find the side length of a square with double the area.

Side of original square $a = 5$ cm.

Diagonal $d = 5\sqrt{2} \approx 7.07$ cm.

New square side length = 7.07 cm.

New area = $7.07^2 = 50$ cm², double the original 25 cm².

Practice Set

- Level 1: Find the diagonal of a square with side 3 cm.
- Level 2: Construct a square with double the area of a square with side 7 cm.
- Level 3: Prove that the square on the diagonal has double the area using congruent triangles.

Answer Key

- Level 1: $3\sqrt{2} \approx 4.24$ cm.
- Level 2: Side length $7\sqrt{2} \approx 9.9$ cm.
- Level 3: Use triangle congruence and area addition to show the doubled area.

Quick Reference

Diagonal of square: $d = a\sqrt{2}$

Area of square on diagonal: $d^2 = 2a^2$

Glossary

- **Square:** A quadrilateral with four equal sides and right angles.
- **Diagonal:** A line segment connecting opposite vertices of a polygon.
- **Area:** The measure of the surface enclosed by a shape.

Halving a Square

To construct a square with half the area of a given square, a smaller square is inscribed inside the original by joining the midpoints of its sides, forming a diamond shape.

Formula Derivation

Let the side length of the original square be a. The smaller square formed by joining midpoints has side length $\frac{a}{\sqrt{2}}$.

Area of original square: a^2.

Area of smaller square:

$$\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{a^2}{2}$$

Thus, the smaller square has half the area of the original.

Worked Illustration

For a square of side 4 cm, the smaller square inside has side $\frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.83$ cm.

Its area is $2.83^2 = 8$ cm², half of the original 16 cm².

Solved Example

Given a square of side 6 cm, find the side length of a square with half the area.

Side length of smaller square = $\frac{6}{\sqrt{2}} = 3\sqrt{2} \approx 4.24$ cm.

Area = $4.24^2 = 18$ cm², half of 36 cm².

Practice Set

- Level 1: Find the side length of a square with half the area of a square with side 8 cm.
- Level 2: Prove that the square formed by joining midpoints has half the area of the original.
- Level 3: Using paper folding, construct a square with half the area of a given square.

Answer Key

- Level 1: $\frac{8}{\sqrt{2}} = 4\sqrt{2} \approx 5.66$ cm.
- Level 2: Use congruent triangles and area relations.
- Level 3: Fold corners to center to form the smaller square.

Quick Reference

Side length of half-area square: $\frac{a}{\sqrt{2}}$

Area: half of original a^2

Glossary

- **Midpoint:** The point dividing a line segment into two equal parts.
- **Inscribed Square:** A square drawn inside another shape touching its sides.

Hypotenuse of an Isosceles Right Triangle

In an isosceles right triangle, the two legs are equal, and the hypotenuse is $\sqrt{2}$ times the length of a leg.

Formula Derivation

Let each leg have length a. By the Pythagorean theorem:

$$c^2 = a^2 + a^2 = 2a^2 \implies c = a\sqrt{2}$$

Worked Illustration

For legs of length 1 unit, hypotenuse $c = 1 \times \sqrt{2} = \sqrt{2} \approx 1.414$ units.

Solved Example

Find the hypotenuse of an isosceles right triangle with legs 12 units.

$c = 12 \times \sqrt{2} = \sqrt{288}$

Since $16^2 = 256$ and $17^2 = 289$, c lies between 16 and 17 units.

Practice Set

- Level 1: Find the hypotenuse when legs are 3 units.
- Level 2: Find legs when hypotenuse is 10 units.
- Level 3: Prove the relation $c = a\sqrt{2}$ using coordinate geometry.

Answer Key

- Level 1: $3\sqrt{2} \approx 4.24$ units.
- Level 2: $a = \frac{10}{\sqrt{2}} = 5\sqrt{2} \approx 7.07$ units.
- Level 3: Use coordinates $(0, 0)$, $(a, 0)$, $(0, a)$ and distance formula.

Quick Reference

Hypotenuse: $c = a\sqrt{2}$

Glossary

- **Isosceles Right Triangle:** A right triangle with two equal legs.
- **Hypotenuse:** The side opposite the right angle, longest side.

Combining Two Squares

To combine two squares of side lengths a and b into a larger square whose area is the sum of the two smaller squares, construct a right triangle with legs a and b. The square on the hypotenuse has area $a^2 + b^2$.

Formula Derivation

By the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

The square on the hypotenuse c has area equal to the sum of the areas of the two smaller squares.

Worked Illustration

For squares of sides 3 and 4 units, the hypotenuse is:

$$c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

The larger square has side 5 units and area 25 units^2.

Solved Example

Given squares of sides 5 and 12 units, find the side of the combined square.

$c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ units.

Practice Set

- Level 1: Combine squares of sides 6 and 8 units.
- Level 2: Prove Baudhāyana's theorem using geometric rearrangement.
- Level 3: Given $c = 15$ and $a = 9$, find b.

Answer Key

- Level 1: $c = 10$ units.
- Level 2: Use congruent triangles and rearrangement of areas.
- Level 3: $b = \sqrt{15^2 - 9^2} = \sqrt{225 - 81} = \sqrt{144} = 12$ units.

Quick Reference

Combined square side: $c = \sqrt{a^2 + b^2}$

Glossary

- **Baudhāyana's Theorem:** The square on the hypotenuse equals the sum of squares on the other two sides.
- **Right Triangle:** Triangle with one 90° angle.

Baudhāyana-Pythagoras Theorem

This theorem states that in a right-angled triangle with sides a, b, and hypotenuse c, the relation holds:

$$a^2 + b^2 = c^2$$

Worked Illustration

For a triangle with legs 3 cm and 4 cm, hypotenuse is:

$$c = \sqrt{3^2 + 4^2} = 5 \text{ cm}$$

Solved Example

Find the hypotenuse of a right triangle with legs 5 cm and 12 cm.

$c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ cm.

Practice Set

- Level 1: Find missing side when $a = 8, b = 15$.
- Level 2: Find missing side when $a = 9, c = 15$.
- Level 3: Prove the theorem using geometric rearrangement.

Answer Key

- Level 1: $c = 17$ cm.
- Level 2: $b = \sqrt{15^2 - 9^2} = 12$ cm.
- Level 3: Use congruent triangles and area addition.

Quick Reference

$$a^2 + b^2 = c^2$$

Glossary

- **Hypotenuse:** Longest side opposite right angle.
- **Legs:** The two sides forming the right angle.

Baudhāyana Triples

Baudhāyana triples are integer triples (a, b, c) satisfying $a^2 + b^2 = c^2$. Examples include (3,4,5), (5,12,13), and (7,24,25).

Properties

- Multiplying a triple by a positive integer k yields another triple (ka, kb, kc).
- Primitive triples have no common factor greater than 1.

Worked Illustration

Scaling (3,4,5) by 2 gives (6,8,10), also a Baudhāyana triple.

Solved Example

Check if (9,12,15) is a Baudhāyana triple.

$$9^2 + 12^2 = 81 + 144 = 225$$

$15^2 = 225$

Yes, it is a triple, scaled by 3 from (3,4,5).

Practice Set

- Level 1: List all Baudhāyana triples with sides ≤ 20.
- Level 2: Find 5 scaled versions of (5,12,13).
- Level 3: Prove that scaling preserves the triple property.

Answer Key

- Level 1: (3,4,5), (5,12,13), (6,8,10), (7,24,25), (8,15,17), (9,12,15), (12,16,20)
- Level 2: (10,24,26), (15,36,39), (20,48,52), (25,60,65), (30,72,78)
- Level 3: Use algebraic expansion to show $(ka)^2 + (kb)^2 = (kc)^2$.

Quick Reference

Baudhāyana triple: $a^2 + b^2 = c^2$ with integer sides.

Glossary

- **Primitive Triple:** Triple with no common factor.
- **Scaled Triple:** Multiple of a primitive triple.

Fermat's Last Theorem

Fermat conjectured that the equation $x^n + y^n = z^n$ has no positive integer solutions for $n > 2$. This was proven by Andrew Wiles in 1994.

Significance

This theorem generalizes the Baudhāyana-Pythagoras theorem to higher powers and states no such integer solutions exist beyond squares.

Practice Set

- Level 1: Understand the statement of Fermat's Last Theorem.
- Level 2: Explore why the Pythagorean theorem holds for $n = 2$.
- Level 3: Research the history and proof outline of Fermat's Last Theorem.

Quick Reference

Equation: $x^n + y^n = z^n$ has no positive integer solutions for $n > 2$.

Glossary

- **Exponent:** The power to which a number is raised.
- **Integer Solutions:** Solutions where variables are whole numbers.

Applications of Baudhāyana-Pythagoras Theorem

The theorem is fundamental in geometry and has applications in calculating distances, constructing shapes, and solving real-world problems.

Example Problem

In a lake, a lotus stem is 1 unit above water and its tip touches water 3 units away when bent. Find the depth of the lake.

Let depth be x. The stem length is $x + 1$. Using the theorem:

$$3^2 + x^2 = (x+1)^2$$

$$9 + x^2 = x^2 + 2x + 1$$

$$9 = 2x + 1 \implies x = 4$$

Depth is 4 units.

Practice Set

- Level 1: Find diagonal of square with side 5 cm.
- Level 2: Find missing sides in given right triangles.
- Level 3: Calculate side length of rhombus given diagonals.

Answer Key

- Level 1: $5\sqrt{2} \approx 7.07$ cm.
- Level 2: Use Pythagorean theorem for each triangle.
- Level 3: Side length = $\frac{1}{2}\sqrt{d_1^2 + d_2^2}$.

Quick Reference

Distance formula and Pythagorean theorem applications.

Glossary

- **Rhombus:** A quadrilateral with all sides equal.
- **Diagonal:** Line segment connecting opposite vertices.