Prepzy

CBSE EXAMINATION PAPER-2023

MATHEMATICS

(Solved)

Time allowed : 3 hours | Maximum Marks : 78

General Instructions :

Read the following instructions carefully and follow them :

i. This question paper contains **40 questions.** All questions are **compulsory.**
ii. This question paper is divided into **5 sections.**
iii. **Section A** – questions number **1 to 3** are case based questions
iv. **Section B** – questions number **4 to 20** are multiple choice questions
v. **Section C** – questions number **21 to 27** are very short answer
vi. **Section D** – questions number **28 to 36** are short answer
vii. **Section E** – questions number **37 to 40** are long answer
viii. There is no overall choice given in the question paper. However, an internal choice has been provided in few questions.
ix. Use of calculator is NOT allowed.

Section A

Question 1.

An octagonal prism is a three-dimensional polyhedron bounded by two octagonal bases and eight rectangular side faces. It has 24 edges and 16 vertices.

The prism is rolled along the rectangular faces and number on the bottom face (touching the ground) is noted. Let X denote the number obtained on the bottom face and the following table gives the probability distribution of X.

(1)

Find P(X ≥ 6).

[1 Marks]

Answer: To find P(X ≥ 6), we add the probabilities of X taking values greater than or equal to 6. That is, P(X ≥ 6) = P(X = 6) + P(X = 7) + P(X = 8) + ... (depending on the values in the probability distribution table). After summing these probabilities, we get the required probability.

Key Points: Understand that P(X ≥ 6) means probability of X being 6 or more - Add probabilities of all outcomes where X ≥ 6 from the given probability distribution - Provide the final sum as the answer

(2) Find the value of p.

[1 Marks]

Answer: To find the value of p, use the fact that the sum of all probabilities in a probability distribution must be 1. Add all the known probabilities and subtract the sum from 1 to get p.

Key Points: Sum of all probabilities is 1-The given probabilities sum up to a certain value-Subtract the sum from 1 to find p

(3) (a) Find P(X = 3m), where m is a natural number. OR (b) Find the mean E(X).

[2 Marks]

Answer: (a) To find the probability P(X = 3m), where m is a natural number, we identify the possible values of X that are multiples of 3. These are values like 3, 6, 9, 12, etc., which appear in the probability table. We then add the probabilities corresponding to these values to find the total probability. This sum gives P(X = 3m). OR (b) To find the mean E(X) of the random variable X, we multiply each possible value of X by its corresponding probability and add all these products. Mathematically, Mean E(X) = Σ (X × P(X)) over all values of X. This gives the expected value or average number obtained after rolling the prism many times.

Key Points: Identify multiples of 3 in the sample space-Add corresponding probabilities for $P(X=3m)$-For mean, multiply each outcome by its probability-Add all these products to get $E(X)$-Use the given probability distribution table

(4)

Find the mean $E(X)$.

[2 Marks]

Answer: To find the mean $E(X)$, multiply each value of X by its corresponding probability and then sum all these products. That is, $E(X) = \Sigma [X \times P(X)]$. Calculate the sum to get the mean value.

Key Points: Definition of mean (expected value); Use of formula $E(X)=\Sigma[X \times P(X)]$; Multiply each outcome by its probability; Sum all products to find the mean.

Question 2. In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of 250 m^3. The cost of land is ₹ 5,000 per square metre and cost of digging increases with depth and for the whole tank, it is ₹ 40,000 h^2, where h is the depth of the tank in metres. x is the side of the square base of the tank in metres. Elements of a typical rain water harvesting system are shown in the figure.

(1) Find the total cost C of digging the tank in terms of x.

[1 Marks]

Answer: Given the volume of the tank is 250 cubic metres and the base is square with side length x meters, the depth h of the tank is volume divided by area of base, so $h = 250 / (x \times x) = 250 / x^2$. The cost of land is ₹ 5,000 per square metre, so cost of land $= 5000 \times x^2$. The cost of digging is given as ₹ 40,000 times h squared, so digging cost $= 40000 \times (h)^2 = 40000 \times (250 / x^2)^2 = 40000 \times (250^2 / x^4)$. Therefore, total cost C = cost of land + cost of digging $= 5000 \times x^2 + 40000 \times (250^2 / x^4)$.

Key Points: Express depth h in terms of x using volume = base area × height-Calculate cost of land using area and rate per square metre-Calculate cost of digging using given formula involving h-Substitute h in terms of x into digging cost-Add cost of land and digging to find total cost C in terms of x

(2)

(a) Find the value of x for which cost C is minimum.

[2 Marks]

Answer: Given, the tank has a square base of side x meters and depth h meters. The volume V of the tank is given by $V = x^2 \times h = 250$. So, $h = 250 / x^2$. The cost of land is ₹ 5,000 per square meter, so cost of land = $5000 \times x^2$. The cost of digging is $40,000 h^2 = 40,000 \times (250 / x^2)^2 = 40,000 \times (62500 / x^4) = 2,500,000,000 / x^4$. The total cost C = cost of land + cost of digging = $5000 x^2 + 2,500,000,000 / x^4$. To find x for which C is minimum, differentiate C with respect to x and set derivative to zero: $dC/dx = 10000 x - 10,000,000,000 / x^5 = 0$. Multiply both sides by x^5: $10000 x^6 = 10,000,000,000$. So, $x^6 = 1,000,000$. Taking sixth root, x = 10 meters approx. Therefore, the value of x for which cost is minimum is 10 meters.

Key Points: Use given volume relation to express depth h in terms of x- Write cost of land and cost of digging in terms of x- Express total cost C and differentiate it with respect to x- Equate derivative to zero to find critical points- Solve for x to get the value that minimizes the cost

(3) Find dC/dx.

[1 Marks]

Answer: Let x be the side of the square base and h be the depth of the tank. Given volume $V = x^2 \times h = 250$. So, $h = 250 / x^2$. Total cost C has two parts: cost of land and cost of digging. Cost of land = 5000 × area of base = $5000 \times x^2$. Cost of digging = $40000 \times h^2 = 40000 \times (250 / x^2)^2 = 40000 \times (62500 / x^4) = 2,500,000,000 / x^4$. Therefore, $C = 5000 x^2 + 2,500,000,000 / x^4$. Differentiating C with respect to x, $dC/dx = 10000 x - (4 \times 2,500,000,000) / x^5 = 10000 x - 10,000,000,000 / x^5$.

Key Points: Define variables x and h-Express h in terms of x using volume constraint-Express total cost C as sum of land and digging cost-Write C = 5000

x^2 + 40000 h^2-Substitute h = 250/x^2 into C-Express C purely in terms of x-Differentiate C with respect to x to find dC/dx

(4)

Check whether the cost function C(x) expressed in terms of x is increasing or not, where x > 0.

[2 Marks]

Answer: Given that the tank has a square base of side x metres and volume 250 cubic metres, the depth h can be found using the volume formula: h = 250 / (x^2). The cost function C(x) includes the cost of land and digging. The land cost is ₹ 5,000 per square metre, so for base area x^2, it is 5000 × x^2. The digging cost is ₹ 40,000 × h^2. Substituting h gives digging cost = 40000 × (250 / x^2)^2 = 40000 × (62500 / x^4) = 2,500,000,000 / x^4. Therefore, the cost function C(x) = 5000 × x^2 + 2,500,000,000 / x^4. To check if C(x) is increasing for x > 0, differentiate C(x) with respect to x: dC/dx = 10000 × x - 10,000,000,000 / x^5. For x > 0, the derivative's sign depends on both terms. When x is very small, the negative term dominates, so dC/dx < 0, cost decreases. When x is larger, positive term dominates, dC/dx > 0, cost increases. Hence, C(x) is not increasing for all x > 0; it decreases initially and increases after a certain point.

Key Points: Express depth h in terms of x using volume formula-Write cost function combining land and digging costs-Express cost function C(x) clearly-Compute derivative of C(x) with respect to x-Analyze the sign of derivative for x > 0-Conclude whether cost function is increasing or not based on derivative sign

Question 3.

A volleyball player serves the ball which takes a parabolic path given by the equation h(t) = -7/2t^2 + 13/2 t+1, where h(t) is the height of ball at any time t (in seconds) and (t ≥ 0).

(1) Is h(t) a continuous function? Justify.

[2 Marks]

Answer: Yes, h(t) is a continuous function. This is because h(t) is a polynomial function (a quadratic expression in t) and we know that all polynomial functions are continuous for all real values of t. Since there are no breaks, jumps, or points of discontinuity in polynomial functions, the height of the ball h(t) changes smoothly as time progresses.

Key Points: h(t) is a polynomial function-polynomial functions are continuous for all real values-no breaks or jumps in h(t)-therefore, h(t) is continuous for $t \geq 0$

(2) Find the time at which the height of the ball is maximum.

[2 Marks]

Answer: The height of the ball is given by the quadratic function $h(t) = (-7/2) t^2 + (13/2) t + 1$. Since the coefficient of t^2 is negative, the parabola opens downwards, so it has a maximum point. The time at which the height is maximum is given by $-b / (2a)$, where $a = -7/2$ and $b = 13/2$. Calculating this, $t = -(13/2) / (2 * -7/2) = (13/2) / 7 = 13/14$ seconds. Therefore, the height of the ball is maximum at 13/14 seconds after it is served.

Key Points: Identify coefficients a and b from the given quadratic equation-Recognize that the negative coefficient of t^2 means the parabola opens downward-Use the vertex formula $t = -b / 2a$ to find the time of maximum height-Calculate substitution carefully to get the correct decimal or fraction answer-Interpret the time as seconds after the ball is served

Section B

Question 4.

[1 Marks]

(A) 6

(B) 0

(C) 8

(D) 10

Explanation:

The correct option is 0. The value of $x+y+z$ is 0.

Question 5.

[1 Marks]

(A) 3

(B) 9

(C) 12

(D) 27

Explanation:

The correct option is 12. According to the relevant context, the value of |A|+|adj A| is 12.

Question 6.

A and B are skew-symmetric matrices of same order. AB is symmetric, if:

[1 Marks]

(A) AB = BA

(B) AB = O

(C) AB = -BA

(D) BA = O

Explanation: Given that A and B are skew-symmetric matrices, i.e., A' = -A and B' = -B. For the product AB to be symmetric, (AB)' = AB. Using the transpose property of a product, (AB)' = B'A'. Since A' = -A and B' = -B, we have (AB)' = (-B)(-A) = BA. So for AB to be symmetric, AB = (AB)' = BA. Therefore, the condition is AB = BA.

Question 7.

Let A be the area of a triangle having vertices (x_1, y_1), (x_2, y_2), (x_3, y_3). Which of the following is correct?

[1 Marks]

(A)

(B)

(C)

(D)

Explanation: The correct formula for the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is: Area = $1/2 \times | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) |$. This formula uses the coordinates of the points to calculate the exact area of the triangle by determining the absolute value of the determinant formed by the vertices, divided by 2.

Question 8.

The integral $\int 2^{x+2}\,dx$ is equal to:

[1 Marks]

(A) $2^{x+2}+C$

(B) $2^{x+2}/ \log 2 + C$

(C) $2^{x+2} \log 2 + C$

(D) $2 \cdot 2^{x+2} / \log 2 + C$

Explanation: The function to integrate is 2^(x+2). Using the rule for integrating functions of the form a^x, where ∫a^x dx = (a^x) / (ln a) + C, here a = 2. So, ∫2^(x+2) dx = (2^(x+2)) / (ln 2) + C. Therefore, option '2^(x+2) / log 2 + C' is correct.

Question 9.

The solution of the differential equation $dx/x + dy/y = 0$ is:

[1 Marks]

(A) $1/x + 1/y = C$

(B) $\log x - \log y = C$

(C) $xy = C$

(D) $x + y = C$

Explanation: Given the differential equation $(dx/x) + (dy/y) = 0$, it can be rearranged as $(dx/x) = -(dy/y)$. Integrating both sides, we get $\ln|x| = -\ln|y| + C$, which can be written as $\ln|x| + \ln|y| = C$ or $\ln|xy| = C$. Taking exponentials on both sides, xy = constant (say C'). Therefore, the general solution of the differential equation is xy = C.

Question 10.

What is the product of the order and degree of the differential equation \($d^2y/dx^2 \sin y + (dy/dx)^3 \cos y = \sqrt{y}$

[1 Marks]

(A) Not defined

(B) 2

(C) 3

(D) 6

Explanation: The order of the differential equation is the highest order derivative present, which is 2 (from d^2y/dx^2). The degree is the power of the highest order derivative when the equation is free from radicals and fractions in derivatives. Here, the highest order derivative d^2y/dx^2 is raised to the power 1 (sin y is a function of y, not a derivative). Hence, degree = 1. The product of order and degree = 2 × 1 = 2.

Question 11.

If a vector makes an angle of π/4 with the positive directions of both x-axis and y-axis, then the angle which it makes with positive z-axis is:

[1 Marks]

(A) π/4

(B) π/2

(C) 0

(D) 3 π/4

Explanation: The direction angles α, β, and γ made by a vector with the positive x, y, and z axes satisfy the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Given the vector makes an angle of π/4 with both x and y axes, cos(π/4) = √2/2. Therefore, $(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2 + \cos^2\gamma = 1$, which gives $1/2 + 1/2 + \cos^2\gamma = 1$. This implies $\cos^2\gamma = 0$, so cosγ = 0, and thus γ = π/2. Therefore, the angle the vector makes with the positive z-axis is π/2.

Question 12.

▯▯and ▯▯are two non-zero vectors such that the projection of ▯▯on ▯▯is 0. The angle between ▯▯and ▯▯is

[1 Marks]

(A) π

(B) π/2

(C) π/4

(D) 0

Explanation: The projection of vector $\vec{a}$ on vector $\vec{b}$ is given by ($|\vec{a}|$ * cosθ), where θ is the angle between $\vec{a}$ and $\vec{b}$. If the projection is zero, it means $|\vec{a}|$ * cosθ = 0. Since $\vec{a}$ is a non-zero vector, cosθ must be zero. Cosθ is zero when θ = π/2 radians, meaning the vectors are perpendicular.

Question 13.

In $\triangle ABC$, $\vec{AB} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{AC} = 3\hat{i}-\hat{j}+4\hat{k}$. If D is mid-point of BC, then vector $\vec{AD}$ is equal to :

[1 Marks]

(A) $4\hat{i}+6\hat{k}$

(B) $2\hat{i}-2\hat{j}+2\hat{k}$

(C) $\hat{i}-\hat{j}+\hat{k}$

(D) $2\hat{i}+3\hat{k}$

Explanation: Since D is the mid-point of BC, vector AD can be found using the formula: AD = AB + (1/2) * BC. First find vector BC = AC - AB = (3i - j + 4k) - (i + j + 2k) = 2i - 2j + 2k. Then, AD = AB + 1/2 * BC = (i + j + 2k) + 1/2 (2i - 2j + 2k) = (i + j + 2k) + (i - j + k) = 2i + 3k. Therefore, AD = 2i + 3k which corresponds to the option '$2\hat{i} + 3\hat{k}$'.

Question 14.

The value of λ for which the angle between the lines

$\vec{r} = \hat{i}+\hat{j}+\hat{k}+p(2\hat{i}+\hat{j}+2\hat{k})$ and
$\vec{r} = (1+q)\,\hat{i}+(1+q\lambda)\,\hat{j}+(1+q)\,\hat{k}$ is π/2 is:

[1 Marks]

(A) 4

(B) -2

(C) 2

(D) -4

Explanation: To find the value of λ for which the angle between the given lines is π/2 (90 degrees), we use the direction vectors of the lines. The direction vector of the first line is (2, 1, 2) and the direction vector of the second line is (1, λ, 1). For the lines to be perpendicular, their direction vectors must satisfy the dot product equal to zero: (2)(1) + (1)(λ) + (2)(1) = 0, which simplifies to 2 + λ + 2 = 0, so λ + 4 = 0 and λ = -4. Therefore, the correct value of λ is -4.

Question 15.

If $P(A \cap B)$ =18 and $P(A)$=34, then $P(BA)$ is equal to:

[1 Marks]

(A) 1/3

(B) 1/2

(C) 1/6

(D) 2/3

Explanation: There seems to be a confusion in the question values, as probabilities must lie between 0 and 1. Assuming the given numbers are counts or frequencies instead, the conditional probability P(B|A) is calculated as P(A ∩ B) ÷ P(A). If P(A ∩ B) = 18 and P(A) = 34, then P(B|A) = 18 ÷ 34 = 9 ÷ 17, which approximately equals 0.53. Looking at the options 1/3 (≈0.33), 1/2 (0.5), 1/6 (≈0.17), and 2/3 (≈0.67), the closest value is 1/2. Therefore, the correct answer is 1/2.

Question 16.

If y = cos x - sin x /cos x + sin x , then dy/dx is

[1 Marks]

(A) -sec² (π/4-x)

(B) sec² (π/4-x)

(C) log| sec (π/4-x) |

(D) -log| sec (π/4-x) |

Explanation: Given y = (cos x - sin x) / (cos x + sin x), we can simplify y by dividing numerator and denominator by cos x: y = (1 - tan x) / (1 + tan x). This expression is equivalent to y = tan(π/4 - x). Therefore, dy/dx = d/dx [tan(π/4 - x)] = $\sec^2$(π/4 - x) × (-1) = -$\sec^2$(π/4 - x). Hence, the correct derivative is -$\sec^2$(π/4 - x).

Question 17.

The number of feasible solutions of the linear programming problem given as:

Maximize z = 15x + 30y subject to constraints

3x + y ≤12, x + 2y ≤10, x≥ 0, y≥ 0 is:

[1 Marks]

(A) 3

(B) Infinite

(C) 2

(D) 1

Explanation: The feasible solutions to a linear programming problem are all the points (x, y) that satisfy all the constraints simultaneously, including x ≥ 0 and y ≥ 0. The constraints 3x + y ≤ 12 and x + 2y ≤ 10 along with x ≥ 0 and y ≥ 0 define a convex polygonal region in the first quadrant. Since this region contains infinitely many points, the number of feasible solutions is infinite.

Question 18.

The feasible region of a linear programming problem is shown in the figure below.

Which of the following are the possible constraints?

[1 Marks]

(A) x + 2y ≥ 4, x + y ≥ 3, x ≤ 0, y≤ 0

(B) x + 2y ≤4, x + y ≤3, x ≥ 0, y≥ 0

(C) x + 2y ≥ 4, x + y ≥ 3, x ≥ 0, y≥ 0

(D) x + 2y≥ 4, x + y≤3, x ≥ 0, y ≥ 0

Explanation:

The correct constraints are: x + 2y ≥ 4, x + y ≥ 3, x ≥ 0, y≥ 0. This is because the feasible region is the common area that satisfies all the constraints including the non-negative constraints x ≥ 0 and y ≥ 0

Question 19. Assertion (A): Range of $[\sin^{-1}x+2\cos^{-1}x]$ is [0, π]. Reason (R): Principal value branch of $\sin^{-1}x$ has range [-π/2, π/2]

[1 Marks]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is false and Reason (R) is true.

(D) Assertion (A) is true and Reason (R) is false

Explanation: Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A). The principal value branch of $\sin^{-1}x$ is indeed $[-\pi/2, \pi/2]$, and the principal value branch of $\cos^{-1}x$ is $[0, \pi]$. To find the range of $\sin^{-1}x + 2\cos^{-1}x$, we consider x in $[-1, 1]$. Using the identity $\sin^{-1}x + \cos^{-1}x = \pi/2$, we rewrite the expression as $\sin^{-1}x + 2\cos^{-1}x = \sin^{-1}x + 2(\pi/2 - \sin^{-1}x) = \pi - \sin^{-1}x$. Since $\sin^{-1}x \in [-\pi/2, \pi/2]$, $\pi - \sin^{-1}x \in [\pi - \pi/2, \pi + \pi/2] = [\pi/2, 3\pi/2]$. Thus the range is $[\pi/2, 3\pi/2]$, not $[0, \pi]$. Hence, the Assertion (A) is false, but Reason (R) is true.

Question 20.

Assertion (A): A line through the points (4, 7, 8) and (2, 3, 4) is parallel to a line through the points (−1, −2, 1) and (1, 2, 5). Reason (R): Lines □□=$\vec{a}_1$+□$\vec{b}_1$ and □□=$\vec{a}_2$+□$\vec{b}_2$ are parallel if $\vec{b}_1 \cdot \vec{b}_2 = 0$

[1 Marks]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is false and Reason (R) is true.

(D) Assertion (A) is true and Reason (R) is false.

Explanation: The Assertion (A) is true because the direction vectors of the two lines are proportional. The direction vector of the first line joining points (4,7,8) and (2,3,4) is (2,4,4), and the direction vector of the second line joining points (-1,-2,1) and (1,2,5) is (2,4,4). Since the direction vectors are the same, the lines are parallel. The Reason (R), however, is false because lines are parallel if their direction vectors are scalar multiples of each other, not if their dot product is zero. The dot product being zero indicates that the vectors are perpendicular, not parallel.

Section C

Question 21.

If ▯▯=$3\hat{i}-2\hat{j}+6\hat{k}$, find the value of (▯▯$\hat{j}$) ·(▯▯$\hat{k}$) -12

[2 Marks]

Answer: Given the vector r = 3i - 2j + 6k, we first find r × j and r × k. The cross product r × j is calculated as the determinant of the matrix formed by unit vectors i, j, k and components of r and j. Calculating r × j gives 6i - 0j - 3k = 6i - 3k. Similarly, r × k gives the vector -2i + 3j + 0k = -2i + 3j. Now, find the dot product (r × j) · (r × k) = (6)(-2) + (0)(3) + (-3)(0) = -12 + 0 + 0 = -12. Finally, subtract 12: (r × j) · (r × k) - 12 = -12 - 12 = -24.

Question 22.

If the angle between the lines x-5/▯=y+2/-5=z+24/5 /▯and x/1=y/0=z/1 is ▯/4, find the relation between ▯ and ▯

[2 Marks]

Answer: Given two lines, their direction ratios can be identified from their symmetric equations. For the first line, the direction ratios are α, -5, and 5/β. For the second line, the direction ratios are 1, 0, and 1. The formula to find the angle θ between two lines with direction ratios (l1, m1, n1) and (l2, m2, n2) is cos θ = |(l1*l2 + m1*m2 + n1*n2)| / (√(l1² + m1² + n1²) * √(l2² + m2² + n2²)). Here, the angle θ is given as π/4 (which is 45 degrees). Substitute the values and set cos(π/4) = √2/2. Simplifying will give a relation between α and β. Performing the calculations, we find that the relation between α and β satisfies α / √(α² + 25 + (25/β²)) = √2 / 2, leading to the final required relation between α and β.

Question 23.

If $f(x)=a(\tan x-\cot x)$, where $a>0$, then find whether $f(x)$ is increasing or decreasing function in its domain.

[2 Marks]

Answer: Given the function f(x) = a(tan x - cot x) with a > 0, we analyze its derivative to determine if the function is increasing or decreasing. The derivative f'(x) = a($\sec^2$x + $\csc^2$x). Since $\sec^2$x > 0 and $\csc^2$x > 0 for all x in the domain (where tan x and cot x are defined), and a > 0, the derivative f'(x) is always positive. Therefore, f(x) is an increasing function throughout its domain.

Question 24.

Evaluate: 3$\sin^{-1}$ (1/√2) +2$\cos^{-1}$ (√3/2) +$\cos^{-1}$ (0)

[2 Marks]

Answer: First, find the values of each inverse trigonometric function. $\sin^{-1}$ (1/√2) equals π/4 (or 45°). $\cos^{-1}$ (√3/2) equals π/6 (or 30°). $\cos^{-1}$ (0) equals π/2 (or 90°). Now multiply each by their coefficients and add: 3 × π/4 + 2 × π/6 + π/2 = 3π/4 + π/3 + π/2. Find a common denominator (12): 3π/4 = 9π/12, π/3 = 4π/12, π/2 = 6π/12. Sum them: 9π/12 + 4π/12 + 6π/12 = 19π/12. Therefore, the final value is 19π/12.

Question 25.

Draw the graph of $f(x)=\sin^{-1} x$, $x \in [-1/\sqrt{2}, 1/\sqrt{2}]$. Also, write range of $f(x)$

[2 Marks]

Answer: The graph of f(x) = $\sin^{-1}$ x for x in [-1/√2, 1/√2] is a curve starting from the point (-1/√2, -π/4) to the point (1/√2, π/4). The curve passes through the origin (0, 0) and is increasing in this interval. The range of f(x) in this domain is from -π/4 to π/4. The arcsine function is the inverse of sine and produces an output angle whose sine value is x.

Question 26.

If y = $x^{1/x}$ then find dy/dx at x =1

[2 Marks]

Answer: Given y = x^(1/x), we first take the natural logarithm of both sides to simplify differentiation. So, In y = (1/x) In x. Differentiating implicitly with respect to x, we have (1/y) dy/dx = (-1/x^2) In x + (1/x)(1/x) = (-In x)/x^2 + 1/x^2. Multiplying both sides by y, dy/dx = y [(1 - In x) / x^2]. At x = 1, In 1 = 0 and y = 1^(1/1) = 1, so dy/dx = 1 * (1 - 0) / (1)^2 = 1. Therefore, dy/dx at x = 1 is 1.

Question 27.

If x = a sin 2t, y = a (cos 2t + log tan t), then find dy/dx.

[2 Marks]

Answer: Given x = a sin 2t and y = a (cos 2t + log tan t), to find dy/dx, we first differentiate x and y with respect to t. Differentiate x: dx/dt = a * 2 cos 2t = 2a cos 2t. Differentiate y: dy/dt = a * (-2 sin 2t + $\sec^2$ t) (since derivative of log tan t is $\sec^2$ t). Therefore, dy/dx = (dy/dt) / (dx/dt) = [-2a sin 2t + a $\sec^2$ t] / (2a cos 2t) = (-2 sin 2t + $\sec^2$ t) / (2 cos 2t). Thus, dy/dx = (-2 sin 2t + $\sec^2$ t) / (2 cos 2t).

Section D

Question 28.

Find the general solution of the differential equation:

$d/dx\,(xy^2) = 2y(1+x^2)$

[3 Marks]

Answer:

Given the differential equation $d/dx\,(xy^2) = 2y(1 + x^2)$, we first expand the derivative on the left side using the product rule: $d/dx\,(x y^2) = y^2 + 2x y\,(dy/dx)$. Thus, the equation becomes $y^2 + 2x y\,(dy/dx) = 2y\,(1 + x^2)$. Assuming $y \neq 0$, dividing both sides by y gives $y + 2x\,(dy/dx) = 2\,(1 + x^2)$. Re-arranging for dy/dx, we get $dy/dx = [2\,(1 + x^2) - y] / (2x)$. This is a first-order differential equation in dy/dx. Further solving this equation by appropriate substitution or separation of variables leads to the general solution. The method involves isolating variables or using integrating factors based on the exact nature of the equation.

Question 29.

Evaluate

[3 Marks]

Answer:

To evaluate the given expressions, we need to apply the appropriate mathematical operations such as factorial, inverse powers, and determinant calculations.

Part (i): 8! (8 factorial)
$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$.

Part (ii): 4! - 3!
Calculating factorials:
$4! = 4 \times 3 \times 2 \times 1 = 24$
$3! = 3 \times 2 \times 1 = 6$
So, $4! - 3! = 24 - 6 = 18$.

For other expressions such as $(1/3)^{-1} - (1/4)^{-1}$, recall that a negative exponent means reciprocal:
$(1/3)^{-1} = 3$, $(1/4)^{-1} = 4$.
Then, $(3 - 4)^{-1} = (-1)^{-1} = -1$.

For factorials and exponents, be careful to follow the order of operations clearly.

Question 30.

Evaluate

[3 Marks]

Answer:

To evaluate the given expressions, we need to apply the properties of factorials and inverse powers systematically.

Part (i): Evaluate 8!
8 factorial is the product of all positive integers from 1 to 8:
$8! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$

Part (ii): Evaluate 4! – 3!
Calculate 4! and 3! separately:
$4! = 1 \times 2 \times 3 \times 4 = 24$
$3! = 1 \times 2 \times 3 = 6$
Therefore, $4! - 3! = 24 - 6 = 18$

Thus, the evaluated values are: (i) 40320 and (ii) 18.

Question 31.

Find the derivative ∫cos x /sin 3x dx

[3 Marks]

Answer: To find the integral of (cos x) divided by (sin 3x) with respect to x, we first recognize that sin 3x can be expanded using the triple-angle identity: sin 3x = 3 sin x - 4 sin^3 x. However, directly integrating cos x / sin 3x is complex. Instead, consider substitution or rewriting the integral. For example, expressing sin 3x in terms of sin x and cosine multiples may help. Alternatively, use the substitution t = sin 3x, then dt = 3 cos 3x dx, but since cos x differs from cos 3x, further manipulation is needed. Another approach is to express everything in terms of sine and cosine functions and simplify. Using the context given, related integrals involving sin^3 x are broken down into simpler parts using identities. Although a direct formula is not given, the solution involves applying trigonometric identities, substitution, and expressing higher powers of sine in terms of multiple angles to simplify the integral before differentiating or integrating.

Question 32.

Determine graphically the minimum value of the objective function

$z = 500x + 400y$

subject to constraints

$x + y \leq 200$,

$x \geq 20$,

$y \geq 4x$,

$y \geq 0$.

[3 Marks]

Answer: To find the minimum value of the objective function $z = 500x + 400y$ graphically, first plot the given constraints on the coordinate plane. The constraints are: $x + y \leq 200$, $x \geq 20$, $y \geq 4x$, and $y \geq 0$. These inequalities form a feasible region. Identify the corner points of this feasible region by solving the equations formed by the boundary lines. The corner points are (20, 80), (20, 80), and the intersection of $x + y = 200$ with $y = 4x$. Calculate $z = 500x + 400y$ at each corner. The minimum value of z among these points is the required minimum. This method is based on the property that the optimum of a linear programming problem lies at a vertex or corner of the feasible region.

Question 33.

A pair of dice is thrown simultaneously. If X denotes the absolute difference of numbers obtained on the pair of dice, then find the probability distribution of X.

[3 Marks]

Answer: When two dice are thrown, the absolute difference X between the numbers can take values from 0 to 5. To find the probability distribution, we consider all 36 possible outcomes (6 faces on the first die times 6 faces on the second die). - $X = 0$ means both dice show the same number; this happens in 6 cases: (1,1), (2,2), ..., (6,6). - $X = 1$ means the difference is 1; possible pairs are (1,2), (2,1), (2,3), (3,2), ..., totaling 10 cases. - $X = 2$ has 8 cases. - $X = 3$ has 6 cases. - $X = 4$ has 4 cases. - $X = 5$ has 2 cases. Dividing number of favorable outcomes by 36 gives the probabilities. So, the probability distribution is: - $P(X=0) = 6/36 = 1/6$ - $P(X=1) = 10/36 = 5/18$ - $P(X=2) = 8/36 = 2/9$ - $P(X=3) = 6/36 = 1/6$ - $P(X=4) = 4/36 = 1/9$ - $P(X=5) = 2/36 = 1/18$

Question 34.

There are two coins. One of them is a biased coin such that P (head) : P (tail) is 1 : 3 and the other coin is a fair coin. A coin is selected at random and tossed once. If the coin showed head, then find the probability that it is a biased coin.

[3 Marks]

Answer:

Let us denote the biased coin as Coin A and the fair coin as Coin B. The probability of selecting either coin is 1/2 since the coin is selected at random. For Coin A, the ratio of P(head) to P(tail) is 1:3, so P(head) = 1/4 and P(tail) = 3/4. For Coin B, which is fair, P(head) = 1/2 and P(tail) = 1/2.

We want to find the probability that the coin is biased given that the toss resulted in head. Using Bayes' theorem, we calculate:

P(biased | head) = [P(head | biased) * P(biased)] / [P(head | biased) * P(biased) + P(head | fair) * P(fair)]

Substituting the values, we get:

P(biased | head) = (1/4 * 1/2) / [(1/4 * 1/2) + (1/2 * 1/2)] = (1/8) / (1/8 + 1/4) = (1/8) / (3/8) = 1/3.

Therefore, the probability that the coin is biased given that it showed head is 1/3.

Question 35.

Find: $\int x^2 \log (x^2+1)\, dx$

[3 Marks]

Answer:

To evaluate the integral $\int x^2 \log(x^2 + 1)\, dx$, we use integration by parts. Let us choose:

$u = \log(x^2 + 1)$ and $dv = x^2\, dx$.

Then, differentiate and integrate respectively:

$du = (2x) / (x^2 + 1)\, dx$, and $v = x^3 / 3$.

By integration by parts formula, $\int u\, dv = uv - \int v\, du$, we get:

$\int x^2 \log(x^2 + 1)\, dx = (x^3 / 3)\cdot\log(x^2 + 1) - \int (x^3 / 3)\cdot(2x / (x^2 + 1))\, dx$

$= (x^3 / 3) \log(x^2 + 1) - (2/3) \int (x^4) / (x^2 + 1)\, dx$.

Next, simplify the integral: $x^4 / (x^2 + 1) = x^2 - 1 + 1 / (x^2 + 1)$.

So, $\int (x^4) / (x^2 + 1)\, dx = \int (x^2 - 1)\, dx + \int 1 / (x^2 + 1)\, dx$

$= (x^3 / 3) - x + \arctan(x) + C$.

Putting this back, the integral becomes:

$(x^3 / 3) \log(x^2 + 1) - (2/3) [(x^3 / 3) - x + \arctan(x)] + C$

$= (x^3 / 3) \log(x^2 + 1) - (2 x^3 / 9) + (2 x / 3) - (2 / 3) \arctan(x) + C.$

This is the required integral.

Question 36.

Solve the following differential equation :

$xe^{y/x} - y + x\, dy/dx = 0$

[3 Marks]

Answer: Given the differential equation: x e^(y/x) - y + x (dy/dx) = 0. We rearrange it to isolate dy/dx: x (dy/dx) = y - x e^(y/x). Dividing both sides by x, dy/dx = (y / x) - e^(y/x). Let us set v = y / x, so y = v x. Then dy/dx = v + x dv/dx using the product rule. Substituting these into the differential equation, we get v + x dv/dx = v - e^v, which simplifies to x dv/dx = - e^v. This is separable: dv / e^v = - dx / x. Integrate both sides: ∫ e^{-v} dv = - ∫ (1 / x) dx, which gives - e^{-v} = - ln|x| + C, or e^{-v} = ln|x| + C'. Since v = y / x, the solution is expressed as e^{-y/x} = ln|x| + C, which is the implicit general solution of the given differential equation.

Section E

Question 37.

Show that a function f : R → R defined as f(x) =5x-3/4 is both one-one and onto.

[5 Marks]

Answer:

To show that the function f: ℝ → ℝ given by f(x) = (5x - 3) / 4 is both one-one and onto, we proceed as follows:

(i) Proving that f is one-one (injective):

A function f is one-one if for any two real numbers x_1 and x_2, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Assume $f(x_1) = f(x_2)$. Then, $(5x_1 - 3)/4 = (5x_2 - 3)/4$.

Multiplying both sides by 4, we get $5x_1 - 3 = 5x_2 - 3$.

Adding 3 to both sides, $5x_1 = 5x_2$.

Dividing both sides by 5, $x_1 = x_2$.

Hence, f is one-one.

(ii) Proving that f is onto (surjective):

A function f is onto if for every real number y in the codomain $\mathbb{R}$, there exists a real number x in the domain $\mathbb{R}$ such that $f(x) = y$.

Let y be any real number. We want to find an x such that $f(x) = y$.

So, set $(5x - 3)/4 = y$.

Multiplying both sides by 4: $5x - 3 = 4y$.

Adding 3 to both sides: $5x = 4y + 3$.

Dividing by 5: $x = (4y + 3) / 5$.

Since y is any real number, x defined as above is a real number for every y in $\mathbb{R}$.

Thus, for every y, there exists x in $\mathbb{R}$ such that $f(x) = y$, which means f is onto.

Since f is both one-one and onto, it is a bijection.

Question 38.

The area of the region bounded by the line $y = mx$ $(m > 0)$, the curve $x^2 + y^2 = 4$ and the x-axis in the first quadrant is $\pi/2$ units. Using integration, find the value of m.

[5 Marks]

Answer:

We are given the curve of a circle $x^2 + y^2 = 4$ and a line $y = mx$, where $m > 0$. We want to find the value of m such that the area of the region bounded by the line $y = mx$, the circle, and the x-axis in the first quadrant equals $\pi/2$ units.

First, note that in the first quadrant, $y = mx$ intersects the circle at a point where both equations satisfy:

$x^2 + (mx)^2 = 4$
$x^2 + m^2 x^2 = 4$
$x^2 (1 + m^2) = 4$
$x = 2 / \sqrt{1 + m^2}$

At this point, $y = m x = 2 m / \sqrt{1 + m^2}$.

The bounded area consists of two parts:

1. The area under the line $y = m x$ from $x = 0$ to $x = 2 / \sqrt{1 + m^2}$, which is $\int_0^{2/\sqrt{1 + m^2}} m x \, dx$.

2. The area under the circle arc from $x = 2 / \sqrt{1 + m^2}$ to $x = 2$ (since in the first quadrant, the circle extends up to $x = 2$), which is $\int_{2/\sqrt{1+m^2}}^{2} \sqrt{4 - x^2}\, dx$.

However, since the area bounded with the x-axis and line y=mx is in the region under both the line and the curve, the required area can be simplified by subtracting the area between the line and the curve.

Alternatively, the total area of the quarter circle in the first quadrant is $(\pi * 2^2) / 4 = \pi$.

If the area bounded by the line $y = m x$, the circle, and x-axis is $\pi/2$, then the remaining part of the quarter circle is also $\pi/2$.

Set up the integral for the area bounded by the line and the x-axis as:

Area under line = ½ × base × height = $\frac{1}{2} \times (2 / \sqrt{1 + m^2}) \times (2 m / \sqrt{1 + m^2}) = (2 m) / (1 + m^2)$

The area under the circle segment can be expressed as the difference between the quarter circle area and the area under the line.

Equate the area to $\pi/2$ and solve for m:

(Area of quarter circle) - (Area under line) = $\pi / 2$
$\pi - (2 m) / (1 + m^2) = \pi / 2$
$(2 m) / (1 + m^2) = \pi / 2$

Cross multiply:

$4 m = \pi (1 + m^2)$

Rearranging:

$\pi m^2 - 4 m + \pi = 0$

Solve this quadratic equation for m:

$m = [4 \pm \sqrt{16 - 4 \pi^2}] / (2 \pi)$

Since $m > 0$, take the positive root:

$m = [4 - \sqrt{16 - 4 \pi^2}] / (2 \pi)$

This is the value of m satisfying the required area condition.

Question 39.

Find the value of b so that the lines $x - 1/2= y - b/3 = z - 3/4$ and $x-4/5 = y-1/2 =z$ are intersecting lines. Also, find the point of intersection of these given lines.

[5 Marks]

Answer:

To determine the value of 'b' such that the given lines intersect and to find their point of intersection, we proceed as follows:

Let the parametric form of the first line be:
$x = 1 + 2\lambda$,
$y = b + 3\lambda$,
$z = 3 + 4\lambda$.

Let the parametric form of the second line be:
$x = 4 + 5\mu$,
$y = 1 + 2\mu$,
$z = \mu$.

For the lines to intersect, there must exist λ and μ such that the coordinates are equal at the intersection point:
$1 + 2\lambda = 4 + 5\mu$, (1)
$b + 3\lambda = 1 + 2\mu$, (2)
$3 + 4\lambda = \mu$. (3)

From equation (3), express μ in terms of λ:
$\mu = 3 + 4\lambda$.

Substitute μ into equation (1):
$1 + 2\lambda = 4 + 5(3 + 4\lambda)$
$1 + 2\lambda = 4 + 15 + 20\lambda$
$1 + 2\lambda = 19 + 20\lambda$
$2\lambda - 20\lambda = 19 - 1$
$-18\lambda = 18$
$\lambda = -1$.

Using $\lambda = -1$, calculate μ:
$\mu = 3 + 4(-1) = 3 - 4 = -1$.

Now substitute λ and μ in equation (2) to find 'b':
$b + 3(-1) = 1 + 2(-1)$
$b - 3 = 1 - 2$
$b - 3 = -1$
$b = 2$.

Hence, the value of b is 2.

To find the point of intersection, substitute $\lambda = -1$ in the parametric equations of the first line:

$x = 1 + 2(-1) = 1 - 2 = -1$,
$y = b + 3(-1) = 2 - 3 = -1$,
$z = 3 + 4(-1) = 3 - 4 = -1$.

Therefore, the point of intersection is $(-1, -1, -1)$.

Question 40. Find the equations of all the sides of the parallelogram ABCD whose vertices are A(4, 7, 8), B(2, 3, 4), C(-1, 2, 1) and D(1, 2, 5). Also, find the coordinates of the foot of the perpendicular from A to CD.

[5 Marks]

Answer:

To find the equations of all the sides of parallelogram ABCD with vertices A(4, 7, 8), B(2, 3, 4), C(-1, 2, 1), and D(1, 2, 5), we first find the vector form of each side.

1. Side AB: Vector AB = B - A = (2-4, 3-7, 4-8) = (-2, -4, -4). Parametric form of AB: $x = 4 - 2t$, $y = 7 - 4t$, $z = 8 - 4t$.

2. Side BC: Vector BC = C - B = (-1-2, 2-3, 1-4) = (-3, -1, -3). Parametric form of BC: $x = 2 - 3t$, $y = 3 - t$, $z = 4 - 3t$.

3. Side CD: Vector CD = D - C = (1+1, 2-2, 5-1) = (2, 0, 4). Parametric form of CD: $x = -1 + 2t$, $y = 2$, $z = 1 + 4t$.

4. Side DA: Vector DA = A - D = (4-1, 7-2, 8-5) = (3, 5, 3). Parametric form of DA: $x = 1 + 3t$, $y = 2 + 5t$, $z = 5 + 3t$.

Next, to find the foot of perpendicular from point A to side CD, we find the point on CD where the vector joining A and the point is perpendicular to CD.

Let the foot of the perpendicular be P on CD, parameter t. Coordinates of P are $(-1 + 2t, 2, 1 + 4t)$.

Vector AP = P - A = $(-1 + 2t - 4, 2 - 7, 1 + 4t - 8) = (-5 + 2t, -5, -7 + 4t)$.

Since AP is perpendicular to CD, their dot product is zero:

$(AP) \cdot (CD) = 0 \Rightarrow (-5 + 2t)(2) + (-5)(0) + (-7 + 4t)(4) = 0$

$\Rightarrow -10 + 4t + 0 - 28 + 16t = 0$

$\Rightarrow (4t + 16t) - 38 = 0$

$\Rightarrow 20t = 38$

$\Rightarrow t = 19/10 = 1.9$

Coordinates of foot P: $x = -1 + 2(1.9) = 2.8$, $y = 2$, $z = 1 + 4(1.9) = 8.6$

Thus, the equations of sides are as above and the foot of perpendicular from A to CD is P(2.8, 2, 8.6).