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Introduction to Sequences

Sequences are ordered lists of numbers where each number is called a term. They represent patterns found in nature, art, music, and many other contexts. Understanding sequences helps us predict subsequent terms and solve real-life problems.

Concept Explanation

Sequences can be finite or infinite. Examples include natural numbers (1, 2, 3, ...), odd numbers (1, 3, 5, ...), triangular numbers (1, 3, 6, 10, ...), and square numbers (1, 4, 9, 16, ...). Each term in these sequences follows a specific pattern or rule.

Formula Derivation

Triangular numbers are sums of natural numbers up to that term:
$t_n = 1 + 2 + 3 + ... + n = {}^{n(n+1)}/_{2}$

Square numbers can be expressed as sums of odd numbers:

$n^2 = 1 + 3 + 5 + ... + (2n-1)$

Worked Illustrations

For triangular numbers: 15 is the 5th term since

$15 = 1 + 2 + 3 + 4 + 5$

For square numbers: 16 is the 4th term since

$16 = 1 + 3 + 5 + 7$

Solved Examples

Find the next three terms of the sequence 1, 4, 7, 10, 13, ...

Solution: The difference between terms is 3, so next terms are 16, 19, 22.

Practice Set

- Level 1: Identify the next three terms in 2, 4, 6, 8, ...
- Level 2: Find the 10th term of the sequence 5, 10, 15, 20, ...
- Level 3: Prove that the sum of the first n odd numbers is n^2

Answer Key

- Level 1: 10, 12, 14
- Level 2: 50
- Level 3: Use induction or sum of arithmetic series of odd numbers.

Quick Reference

- Sequence: Ordered list of terms.
- Term: Each element in the sequence.
- Triangular number formula:
  $t_n = {}^{n(n+1)}/_{2}$
- Square number formula:
  $n^2 = 1 + 3 + 5 + ... + (2n-1)$

Glossary

- **Sequence:** An ordered list of numbers.
- **Term:** An individual element of a sequence.
- **Triangular Number:** Sum of first n natural numbers.
- **Square Number:** Sum of first n odd numbers.

Explicit and Recursive Formulas for Sequences

Sequences can be described using explicit formulas or recursive formulas.

Concept Explanation

An explicit formula expresses the nth term directly in terms of n, e.g.,
$t_n = 2n - 1$

A recursive formula defines each term using previous terms, e.g.,
$t_n = t_{(n-1)} + 3$

with initial term
$t_1 = 1$.

Formula Derivation

For the sequence 1, 4, 7, 10, 13, ... the explicit formula is $t_n = 3n - 2$. The recursive formula is

$t_1 = 1, t_n = t_{(n-1)} + 3$

for n greater or equal to 2.

Worked Illustrations

Using explicit formula
$u_n = 2n - 1$, find u_1, u_2, u_3:

- $u_1 = 2 \times 1 - 1 = 1$
- $u_2 = 2 \times 2 - 1 = 3$
- $u_3 = 2 \times 3 - 1 = 5$

Solved Examples

Find the first four terms of the sequence defined by
$u_1 = 1, u_n = 2u_{(n-1)} + 3$

for n greater or equal to 2.

Solution:

- $u_2 = 2 \times 1 + 3 = 5$
- $u_3 = 2 \times 5 + 3 = 13$
- $u_4 = 2 \times 13 + 3 = 29$

Practice Set

- Level 1: Write explicit formula for the sequence 3, 6, 9, 12, ...
- Level 2: Find the 100th term of the sequence defined by $s_n = 5n - 2$.
- Level 3: Given $t_1 = 3, t_n = t_{(n-1)}(t_{(n-1)} - 1)$, find t_4 .

Answer Key

- Level 1:
  $t_n = 3n$
- Level 2:
  $s_{100} = 5 \text{ times } 100 - 2 = 498$
- Level 3:
  $t_2 = 3 \text{ times } 2 = 6, t_3 = 6 \text{ times } 5 = 30, t_4 = 30 \text{ times } 29 = 870$

Quick Reference

- Explicit formula:
  t_n in terms of n
- Recursive formula:
  t_n in terms of $t_{(n-1)}$

Glossary

- **Explicit Formula:** Formula to find the nth term directly.
- **Recursive Formula:** Formula to find the nth term using previous terms.

Arithmetic Progressions

An arithmetic progression (AP) is a sequence where the difference between consecutive terms is constant.

Concept Explanation

In an AP, each term after the first is obtained by adding a fixed number called the common difference (d) to the previous term.

Formula Derivation

The nth term of an AP is given by:

$t_n = a + (n - 1)d$

where a is the first term and d is the common difference.

Worked Illustrations

For the sequence 1, 5, 9, 13, 17, ...:

- First term $a = 1$
- Common difference $d = 4$
- nth term $t_n = 1 + (n - 1) \times 4 = 4n - 3$

Solved Examples

Find the 10th term of the AP 2, 5, 8, 11, ...

Solution:

- First term a = 2
- Common difference d = 3
- t_10 = 2 + (10 - 1) times 3 = 2 + 27 = 29

Practice Set

- Level 1: Find the 15th term of 1/2, 5/2, 9/2, ...
- Level 2: Write the recursive formula for the AP 3, 7, 11, 15, ...
- Level 3: A taxi charges 5200 fixed plus 540 per km. Write the fare sequence and find fare for 10 km.

Answer Key

- Level 1: t_15 = $^1/_2$ + (15 - 1) times 2 = 27.5
- Level 2: t_1 = 3, t_n = t_(n-1) + 4
- Level 3: Fare sequence: 240, 280, 320, ...; 10 km fare = 5600

Quick Reference

- AP general term: t_n = a + (n - 1)d
- Common difference: d = t_2 - t_1

Glossary

- **Arithmetic Progression:** Sequence with constant difference.
- **Common Difference:** Fixed difference between terms.

Sum of First n Natural Numbers and Triangular Numbers

We derive the formula for the sum of the first n natural numbers and relate it to triangular numbers.

Concept Explanation

Sum of first n natural numbers is denoted by S_n = 1 + 2 + 3 + ... + n .

Formula Derivation

Using the pairing method:

S_n = 1 + 2 + 3 + ... + n
S_n = n + (n-1) + (n-2) + ... + 1
2S_n = (n+1) + (n+1) + ... + (n+1) = n(n+1)
Rightarrow S_n = $^{n(n+1)}/_{2}$

Worked Illustrations

Sum of first 10 natural numbers:

S_10 = $^{10 \text{ times } 11}/_{2}$ = 55

Solved Examples

Find the 10th, 17th, and 80th triangular numbers.

Solution:

- $t_{10} = {}^{10 \text{ times } 11}/_{2} = 55$
- $t_{17} = {}^{17 \text{ times } 18}/_{2} = 153$
- $t_{80} = {}^{80 \text{ times } 81}/_{2} = 3240$

Practice Set

- Level 1: Find sum of first 20 natural numbers.
- Level 2: Find sum of numbers from 25 to 58.
- Level 3: Prove the formula for sum of first n natural numbers using induction.

Answer Key

- Level 1: 210
- Level 2: 1411
- Level 3: Use mathematical induction steps.

Quick Reference

- Sum of first n natural numbers: $S_n = {}^{n(n+1)}/_{2}$
- Triangular number: $t_n = S_n$

Glossary

- **Sum of Natural Numbers:** Total of first n natural numbers.
- **Triangular Number:** Sum of first n natural numbers.

Geometric Progressions

A geometric progression (GP) is a sequence where each term after the first is obtained by multiplying the previous term by a fixed number called the common ratio.

Concept Explanation

In a GP, the ratio between consecutive terms is constant.

Formula Derivation

The nth term of a GP is given by:

$t_n = a r^{(n-1)}$

where a is the first term and r is the common ratio.

Worked Illustrations

For the sequence 3, 6, 12, 24, ...:

- First term $a = 3$
- Common ratio $r = 2$
- nth term $t_n = 3$ times $2^{(n-1)}$

Solved Examples

Check if 1, 3, 9, 27, 81, ... is a GP and find the common ratio.

Solution: Ratio between terms is 3, so it is a GP with $r = 3$.

Practice Set

- Level 1: Find the 5th term of 2, 10, 50, 250, ...
- Level 2: Find the nth term of 4, 8/3, 16/9, 32/27, ...
- Level 3: Find recursive formula for t_n = 3 times 10^(n-1) .

Answer Key

- Level 1: t_5 = 2 times 5^4 = 1250
- Level 2: t_n = 4 times (2/3)^(n-1)
- Level 3: t_1 = 3, t_n = 10 t_(n-1)

Quick Reference

- GP general term: t_n = a r^(n-1)
- Common ratio: r = t_n / t_(n-1)

Glossary

- **Geometric Progression:** Sequence with constant ratio.
- **Common Ratio:** Fixed multiplier between terms.

Fractals and the Sierpi□44ski Triangle

Fractals are patterns that repeat at different scales. The Sierpi□44ski triangle is a famous fractal formed by repeatedly removing central triangles from an equilateral triangle.

Concept Explanation

The number of black triangles at stage n is $t_n = 3^n$. The area of the black region at stage n is $s_n = (3/4)^n$.

Formula Derivation

Each stage multiplies the number of black triangles by 3 and the area by $3/4$.

Worked Illustrations

Stages 0 to 3 have black triangles: 1, 3, 9, 27.

Areas at stages 0 to 3: $1, 3/4, (3/4)^2, (3/4)^3$.

Solved Examples

Find the number of black triangles and area at stage 5.

Solution:

- Number of black triangles: $3^5 = 243$
- Area: $(3/4)^5$

Practice Set

- Level 1: Find number of black triangles at stage 4.
- Level 2: Find area of black region at stage 6.
- Level 3: Explain why the area approaches zero as n approaches infinity.

Answer Key

- Level 1: 81
- Level 2: (3/4)^6
- Level 3: Because (3/4)^n decreases exponentially.

Quick Reference

- Number of black triangles: t_n = 3^n
- Area of black region: s_n = (3/4)^n

Glossary

- **Fractal:** A pattern repeating at different scales.
- **Sierpi▯44ski Triangle:** A fractal formed by recursive removal of triangles.