Relations

A relation R from a non-empty set A to another non-empty set B is defined as a subset of the Cartesian product $A \times B$. Formally,

$$R \subseteq A \times B = \{(a, b) : a \in A, b \in B\}$$

Thus, any subset of $A \times B$ is a relation from A to B.

Note: If A and B are finite sets with p and q elements respectively, then $n(A \times B) = pq$. The total number of relations from A to B is the number of subsets of $A \times B$, which is 2^{pq}.

Domain, Range and Co-domain of a Relation

Domain: The domain of a relation R from A to B is the set of all elements $a \in A$ such that there exists $b \in B$ with $(a, b) \in R$. Formally,

$$\text{Dom}(R) = \{a \in A : \exists b \in B, (a, b) \in R\}$$

Range: The range of R is the set of all elements $b \in B$ such that there exists $a \in A$ with $(a, b) \in R$. Formally,

$$\text{Range}(R) = \{b \in B : \exists a \in A, (a, b) \in R\}$$

Co-domain: The co-domain of R is the set B itself.

Types of Relations

- **Empty Relation:** $R = \emptyset$.
- **Universal Relation:** $R = A \times B$.
- **Identity Relation:** Defined on A as $I_A = \{(a, a) : a \in A\}$.
- **Reflexive Relation:** R on A is reflexive if $(a, a) \in R$ for all $a \in A$.
- **Symmetric Relation:** R on A is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
- **Transitive Relation:** R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ for all $a, b, c \in A$.
- **Equivalence Relation:** A relation that is reflexive, symmetric, and transitive.

Worked Example 1

Let $A = \{1, 2, 3, 7\}$, $B = \{3, 6\}$, and define $R = \{(a, b) : a < b\}$. Find domain, range, and co-domain.

Solution:

$R = \{(1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\}$

Domain: $\{1, 2, 3\}$

Range: $\{3,6\}$

Co-domain: $B=\{3,6\}$

Worked Example 2

Check if the relation $R=\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3)\}$ on $A=\{1,2,3\}$ is reflexive.

Solution:

Check if $(a,a)\in R$ for all $a\in A$:

$(1,1),(2,2),(3,3)\in R$, so R is reflexive.

Practice Set

Level 1 – Easy

- Define the relation R on $A=\{1,2\}$ and $B=\{3,4\}$ as $R=\{(1,3)\}$. Find domain, range, and co-domain.
- Is the empty relation on $A=\{1,2,3\}$ reflexive?

Level 2 – Moderate

- Given $A=\{1,2,3\}$, check if $R=\{(1,2),(2,1),(2,3),(3,2)\}$ is symmetric.
- Show that the universal relation on $A=\{1,2\}$ is reflexive.

Level 3 – Challenging

- Prove that the relation $R = \{(a, b) : a - b \text{ is even}\}$ on integers is an equivalence relation.
- Given $A = \{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, verify if R is transitive.

Answer Key

Level 1

- Domain: $\{1\}$, Range: $\{3\}$, Co-domain: $\{3, 4\}$
- No, empty relation is not reflexive as $(a, a) \notin R$ for all $a \in A$.

Level 2

- Yes, R is symmetric because for every $(a, b) \in R$, $(b, a) \in R$.
- Universal relation contains all pairs including (a, a), so it is reflexive.

Level 3

- Reflexive: $a - a = 0$ even, Symmetric: if $a - b$ even, then $b - a$ even, Transitive: sum of even numbers is even. Hence equivalence relation.
- Check transitivity: For all $(a, b), (b, c) \in R$, $(a, c) \in R$ holds. Hence R is transitive.

Quick Reference

Relation Type	Definition
Empty Relation	$R = \emptyset$

Universal Relation	$R = A \times B$
Identity Relation	$I_A = \{(a, a) : a \in A\}$
Reflexive	$(a, a) \in R$ for all $a \in A$
Symmetric	$(a, b) \in R \Rightarrow (b, a) \in R$
Transitive	$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$
Equivalence Relation	Reflexive, Symmetric, and Transitive

Glossary

- **Relation:** A subset of $A \times B$.
- **Domain:** Set of first elements in relation pairs.
- **Range:** Set of second elements in relation pairs.
- **Co-domain:** The set B in relation $R \subseteq A \times B$.
- **Reflexive:** Every element relates to itself.
- **Symmetric:** Relation is bidirectional.
- **Transitive:** Relation passes through intermediate elements.
- **Equivalence Relation:** Relation that is reflexive, symmetric, and transitive.

Functions

A function f from a set A to a set B is a special type of relation where each element of A is related to exactly one element of B. Formally, $f \subseteq A \times B$ such that for every $x \in A$, there exists a unique $y \in B$ with $(x, y) \in f$. The element y is called the image of x under f, denoted $f(x)$.

Difference Between Relation and Function

Property	Function	Relation
Existence	Every $x \in A$ has at least one $y \in B$ with $(x, y) \in f$	Not necessarily

Uniqueness	Each $x \in A$ has exactly one $y \in B$	Can have multiple y for same x

Real Valued Function of a Real Variable

If the domain and range of f are subsets of real numbers $\mathbb{R}$, then f is called a real valued function of a real variable.

Common Real Functions

Function	Expression	Domain	Range
Identity	$f(x) = x$	$\mathbb{R}$	$\mathbb{R}$
Modulus	$f(x) = \|x\|$	$\mathbb{R}$	$[0, \infty)$
Greatest Integer	$f(x) = \lfloor x \rfloor$	$\mathbb{R}$	$\mathbb{Z}$
Signum	$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$	$\mathbb{R}$	$\{-1, 0, 1\}$
Exponential	$f(x) = a^x, a > 0, a \neq 1$	$\mathbb{R}$	$(0, \infty)$
Logarithmic	$f(x) = \log_a x, a > 0, a \neq 1$	$(0, \infty)$	$\mathbb{R}$

Types of Functions

- **One-one (Injective) Function:** $f : A \to B$ is injective if $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.
- **Onto (Surjective) Function:** $f : A \to B$ is surjective if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.
- **Bijective Function:** A function that is both injective and surjective.
- **Identity Function:** $I_A : A \to A$ defined by $I_A(x) = x$ for all $x \in A$.
- **Equal Functions:** Two functions f and g are equal if $f(x) = g(x)$ for all x in their domain.

Algorithms to Check Injectivity and Surjectivity

Injectivity

1. Take arbitrary $a, b \in A$.
2. Assume $f(a) = f(b)$.
3. Solve for $a = b$. If true for all a, b, f is injective.

Surjectivity

1. Take arbitrary $b \in B$.
2. Solve $f(x) = b$ for x.
3. If solution $x \in A$ exists for all b, f is surjective.

Worked Example 1

Show that $f : A \to B$ defined by $f(x) = 4x + 7$ is one-one.

Solution:

Assume $f(x_1) = f(x_2)$, then

$$4x_1 + 7 = 4x_2 + 7$$

$$4x_1 = 4x_2$$

$$x_1 = x_2$$

Hence, f is injective.

Worked Example 2

Show that $f : \mathbb{N} \to \mathbb{N}$ defined by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for $x > 2$ is onto but not one-one.

Solution:

Since $f(1) = f(2) = 1$, f is not injective.

For any $y \in \mathbb{N}$, $y \neq 1$, choose $x = y + 1$, then $f(x) = y$. Also $f(1) = 1$. Hence, f is onto.

Practice Set

Level 1 – Easy

- Define a function $f : \{1, 2, 3\} \to \{4, 5\}$ and find its domain, co-domain, and range.
- Is the function $f(x) = 2x + 3$ injective?

Level 2 – Moderate

- Check if $f(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ is onto.
- Show that the function $f(x) = \sin x$ is not one-one.

Level 3 – Challenging

- Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ is bijective.

- Find the inverse of the function $f(x) = \frac{2x-1}{3}$ and verify it is bijective.

Answer Key

Level 1

- Domain: $\{1, 2, 3\}$, Co-domain: $\{4, 5\}$, Range depends on function definition.
- Yes, $f(x) = 2x + 3$ is injective.

Level 2

- $f(x) = x^2$ is not onto $\mathbb{R}$ because negative numbers are not in range.
- $f(x) = \sin x$ is not one-one as $\sin x = \sin(\pi - x)$.

Level 3

- $f(x) = 3x + 2$ is bijective because it is both injective and surjective.
- Inverse: $f^{-1}(y) = \frac{3y+1}{2}$. Verified by composition.

Quick Reference

Function Type	Condition
Injective	$f(a) = f(b) \Rightarrow a = b$
Surjective	$\forall b \in B, \exists a \in A : f(a) = b$
Bijective	Both injective and surjective
Identity	$I_A(x) = x$

Glossary

- **Function:** Relation with unique image for each element in domain.
- **Domain:** Set of inputs.
- **Co-domain:** Set of possible outputs.
- **Range:** Actual set of outputs.
- **Injective:** One-to-one mapping.
- **Surjective:** Onto mapping.
- **Bijective:** Both injective and surjective.
- **Inverse Function:** Function reversing f.