Inverse Trigonometric Functions

Inverse trigonometric functions allow us to find the angle when the value of a trigonometric ratio is known. These functions are the inverses of the standard trigonometric functions: sine, cosine, tangent, cosecant, secant, and cotangent.

Concept Explanation

For a function $f : X \to Y$ that is one-to-one and onto, the inverse function $g : Y \to X$ satisfies $g = f^{-1}$, with domain and range interchanged:

$$\text{Domain of } g = \text{Range of } f, \quad \text{Range of } g = \text{Domain of } f$$

Inverse trigonometric functions are defined by restricting the domain of the original trigonometric functions to make them one-to-one and onto, enabling the existence of inverses.

Formula Derivation

For example, the inverse sine function $y = \sin^{-1} x$ is defined as the inverse of $y = \sin x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus,

$$\sin y = x, \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad x \in [-1, 1]$$

Similarly, inverse cosine and inverse tangent functions are defined with appropriate domain restrictions to ensure invertibility.

Worked Illustrations

Example 1: Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution:

Let $y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$. Then,

$$\sin y = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \implies y = \frac{\pi}{4}$$

Since $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the principal value is $\frac{\pi}{4}$.

Solved Examples

Example 2: Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$.

Solution:

Using the formula for sum of inverse tangents:

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

Let $A = \frac{1}{2}, B = \frac{2}{11}$, then

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}\right) = \tan^{-1}\left(\frac{\frac{15}{22}}{1-\frac{1}{11}}\right) = \tan^{-1}\left(\frac{\frac{15}{22}}{\frac{10}{11}}\right) = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4}$$

Hence proved.

Practice Set

Level 1 – Easy

- Find $\sin^{-1} 0$.
- Evaluate $\cos^{-1} 1$.
- Find the principal value of $\tan^{-1} 1$.

Level 2 – Moderate

- Prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $x \in [-1, 1]$.
- Find $\tan^{-1} 2 + \tan^{-1} 3$.
- Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Level 3 – Challenging

- Show that $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$.
- Find the exact value of $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5}$.
- Prove that $\cos^{-1} x + \cos^{-1} y = \pi$ if $x^2 + y^2 = 1$ and $x, y \in [0, 1]$.

Answer Key

- $\sin^{-1} 0 = 0$
- $\cos^{-1} 1 = 0$
- $\tan^{-1} 1 = \frac{\pi}{4}$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (proved by definition)
- $\tan^{-1} 2 + \tan^{-1} 3 = \tan^{-1}\left(\frac{2+3}{1-2\times3}\right) = \tan^{-1}(-1) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$
- $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
- $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$ (using tangent addition formula)
- $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}$
- $\cos^{-1} x + \cos^{-1} y = \pi$ if $x^2 + y^2 = 1$ (using trigonometric identities)

Quick Reference

Function	Domain	Range (Principal Value)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$

Glossary

- **Domain:** Set of all possible input values for a function.
- **Range:** Set of all possible output values of a function.
- **Principal Value:** The unique value of an inverse trigonometric function chosen from its range to make it a function.
- **One-to-One Function:** A function where each input corresponds to a unique output.
- **Inverse Function:** A function that reverses the effect of the original function.