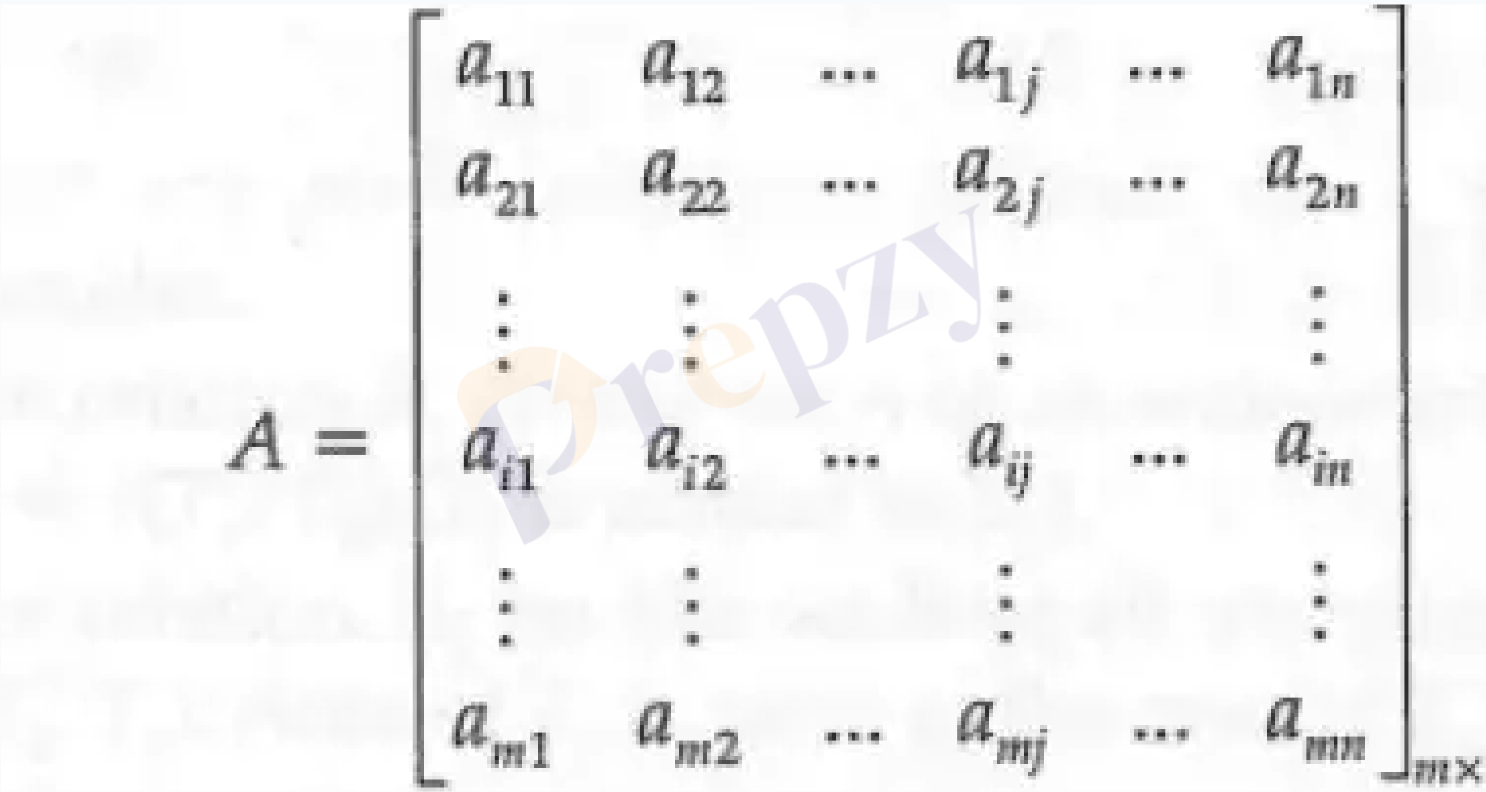
Matrices and Operations

A matrix is an ordered rectangular array of numbers or functions arranged in rows and columns. It is denoted by uppercase letters such as
order $m \times n$ has m rows and n columns, with elements a_{ij} where i is the row number and j is the column number.



The elements are arranged as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Types of Matrices

- **Column matrix:** A matrix with only one column, order $m \times 1$.
- **Row matrix:** A matrix with only one row, order $1 \times n$.
- **Square matrix:** Number of rows equals number of columns, order $n \times n$.
- **Diagonal matrix:** Square matrix where all off-diagonal elements are zero, i.e., $a_{ij} = 0$ for $i \neq j$.
- **Scalar matrix:** Diagonal matrix with all diagonal elements equal to a constant k, i.e., $a_{ij} = k$ if $i = j$, else 0.
- **Identity matrix:** Scalar matrix with $k = 1$, denoted by I_m, where $a_{ij} = 1$ if $i = j$, else 0.

- **Zero matrix:** All elements are zero.
- **Horizontal matrix:** Matrix with more columns than rows, $m < n$.
- **Vertical matrix:** Matrix with more rows than columns, $m > n$.

Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order $m \times n$ are equal if and only if $a_{ij} = b_{ij}$ for all i, j.

Addition of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, their sum $A + B = [a_{ij} + b_{ij}]$ is also a matrix of order $m \times n$. Addition is defined only same order.

Multiplication of a Matrix by a Scalar

For a scalar k and matrix $A = [a_{ij}]$, the product $kA = [ka_{ij}]$ is obtained by multiplying each element of A by k.

Multiplication of Two Matrices

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. The product $C = AB = [c_{ik}]$ is an $m \times p$ matrix where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

Matrix multiplication is defined only if the number of columns of A equals the number of rows of B. Note that matrix multiplication is not com

Properties of Matrix Multiplication

- If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.
- If A and B are square matrices of order n, and I_n is the identity matrix, then $I_n A = AI_n = A$.

Idempotent Matrix

A square matrix A is idempotent if $A^2 = A$.

Transpose of a Matrix

The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^T = [a_{ji}]$ obtained by interchanging rows and columns.

Properties of Transpose

- $(A + B)^T = A^T + B^T$
- $(A^T)^T = A$

- $(kA)^T = kA^T$, where k is a scalar
- $(AB)^T = B^T A^T$
- $(ABC)^T = C^T B^T A^T$

Worked Examples

Example 1

Problem: Find all possible orders of a matrix with 8 elements.

Solution: If a matrix has order $m \times n$, then $mn = 8$. The possible pairs (m, n) are (1,8), (8,1), (2,4), (4,2). Hence, possible orders are $1 \times 8, 8 \times$

Example 2

Problem: Construct a 3×2 matrix $A = [a_{ij}]$ where $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution:

$$
\begin{aligned}
a_{11} &= \frac{1}{2}|1 - 3 \times 1| = \frac{1}{2}|1 - 3| = 1 \\
a_{12} &= \frac{1}{2}|1 - 3 \times 2| = \frac{1}{2}|1 - 6| = \frac{5}{2} \\
a_{21} &= \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2}|2 - 3| = \frac{1}{2} \\
a_{22} &= \frac{1}{2}|2 - 3 \times 2| = \frac{1}{2}|2 - 6| = 2 \\
a_{31} &= \frac{1}{2}|3 - 3 \times 1| = \frac{1}{2}|3 - 3| = 0 \\
a_{32} &= \frac{1}{2}|3 - 3 \times 2| = \frac{1}{2}|3 - 6| = \frac{3}{2}
\end{aligned}
$$

Thus,

$$
A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}
$$

Example 3

Problem: Multiply matrices

$$
A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}
$$

Solution: Since A is 2×2 and B is 2×3, multiplication AB is defined and will be 2×3.

$$AB = \begin{bmatrix} (6\times 2+9\times 7) & (6\times 6+9\times 9) & (6\times 0+9\times 8) \\ (2\times 2+3\times 7) & (2\times 6+3\times 9) & (2\times 0+3\times 8) \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Example 4

Problem: Verify the matrix identity $(AB)^T = B^T A^T$ for given matrices A and B.

Solution: Calculate AB, then $(AB)^T$. Calculate A^T and B^T, then multiply $B^T A^T$. Verify both results are equal.

Practice Set

Level 1 – Easy

- Find the order of a matrix with 12 elements.
- Write a 2×3 zero matrix.
- Add two matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Level 2 – Moderate

- Multiply matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ by scalar 3.
- Find the product of $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$.
- Find the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Level 3 – Challenging

- Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^2 and verify if A is idempotent.
- Find the product AB where $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$.
- Verify the property $(ABC)^T = C^T B^T A^T$ for matrices A, B, C of appropriate orders.

Answer Key

Level 1

- Possible orders for 12 elements: (1,12), (12,1), (2,6), (6,2), (3,4), (4,3).
- Zero matrix 2×3: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- Sum: $\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$.

Level 2

- Scalar multiplication: $\begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$.
- Product: $\begin{bmatrix} 2 & 10 \\ 7 & 19 \end{bmatrix}$.
- Transpose: $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Level 3

- $A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$, so $A^2 \neq A$, not idempotent.
- $AB = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$.
- Verification requires matrix multiplication and transpose; results will be equal.

Quick Reference

Operation	Formula	Condition
Addition	$(A+B)_{ij} = a_{ij} + b_{ij}$	Same order matrices
Scalar Multiplication	$(kA)_{ij} = ka_{ij}$	Any matrix
Matrix Multiplication	$(AB)_{ik} = \sum_j a_{ij}b_{jk}$	Columns of A = Rows of B
Transpose	$(A^T)_{ij} = a_{ji}$	Any matrix
Idempotent	$A^2 = A$	Square matrix

Glossary

- **Matrix:** Rectangular array of elements arranged in rows and columns.
- **Order:** Number of rows and columns of a matrix, denoted $m \times n$.
- **Element:** Individual entry in a matrix, denoted a_{ij}.
- **Scalar:** A single number used to multiply a matrix.
- **Transpose:** Matrix obtained by interchanging rows and columns.
- **Idempotent matrix:** Matrix satisfying $A^2 = A$.

Symmetric, Skew-Symmetric and Invertible Matrices

A square matrix $A = [a_{ij}]$ is:

Symmetric Matrix

If $A^T = A$, i.e., $a_{ij} = a_{ji}$ for all i, j.

For example:

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$$

Skew-Symmetric Matrix

If $A^T = -A$, i.e., $a_{ij} = -a_{ji}$ and all diagonal elements $a_{ii} = 0$.

For example :

$$\begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Orthogonal Matrix

A matrix A is orthogonal if $AA^T = I$, where I is the identity matrix.

Invertible Matrix

A square matrix A is invertible if there exists a matrix B such that $AB = BA = I$, where I is the identity matrix.

Worked Examples

Example 4

Problem: Show that the inverse of a matrix, if it exists, is unique.

Solution: Given matrices A and B such that $AB = BA = I$, B is the inverse of A. If another matrix C also satisfies $AC = CA = I$, then $B =$

Example 4

Let matrix A = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and matrix B = $\begin{bmatrix} 5 & - \\ -2 & 1 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, $A^{-1} = B$ and B is called the inverse of A.

So, A can also be the inverse of B or $B^{-1} = A$.

Uniqueness of Inverse of Matrix

If there exists an inverse of a square matrix
always unique.

Proof: Let A be a square matrix of order $n \times$
us assume matrices B and C be inverses of mat

Now, AB = BA = I, since B is the inverse of matri

Similarly, AC = CA = I

But, B = BI = B(AC) = (BA)C = IC = C

This proves B = C, or B and C are the same ma

Example 5

Problem: Express matrix B as the sum of a symmetric matrix P and a skew-symmetric matrix Q.

Solution:

$$P = \frac{1}{2}(B + B^T), \quad Q = \frac{1}{2}(B - B^T), \quad B = P + Q$$

Calculate B^T, then P and Q accordingly.

EXAMPLE 5

Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Sol. Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let

$$P = \frac{1}{2}(B + B')$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now

$$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

$$P' = P$$

Here, $P' = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let $Q = \frac{1}{2}(B - B')$

$$= \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then

$$Q' = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Here, $Q' = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symme

Now

$$P+Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \ldots \\ \frac{1}{2} & \ldots \\ \frac{5}{2} & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Thus, B is represented as the sum of a a skew symmetric matrix.

Key Facts

- Diagonal elements of a skew-symmetric matrix are zero.
- For any matrix A, AA^T and A^TA are symmetric.

- Any square matrix A can be expressed as $A = P + Q$, where $P = \frac{1}{2}(A + A^T)$ is symmetric and $Q = \frac{1}{2}(A - A^T)$ is skew-symmetric.

Glossary

- **Symmetric matrix:** Matrix equal to its transpose.
- **Skew-symmetric matrix:** Matrix whose transpose equals its negative.
- **Orthogonal matrix:** Matrix whose product with its transpose is the identity matrix.
- **Invertible matrix:** Matrix having a unique inverse such that their product is the identity matrix.