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Determinants, Minors and Co-factors

Determinants are unique scalar values associated with square matrices. For a square matrix $A = [a_{ij}]$ of order m, the determinant is denoted by $|A|$ or $\det(A)$.

Determinant of a 2x2 matrix:

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is

$$|A| = ad - bc$$

Minors

The minor M_{ij} of an element a_{ij} in matrix A is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column from A.

Co-factors

The cofactor C_{ij} of element a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of a_{ij}.

Adjoint of a Square Matrix

For a square matrix $A = [a_{ij}]$, let $B = [C_{ij}]$ be the matrix of cofactors. The transpose of B, denoted B^T, is called the adjoint of A and is written as $\text{adj}(A)$.

For a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the adjoint is

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Worked Example

Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

Step 1: Find the cofactor matrix $B = [C_{ij}]$.

Calculate each cofactor:

- $C_{11} = (-1)^{1+1} \times \det \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = 1 \times (3 \times 5 - 4 \times 0) = 15$
- $C_{12} = (-1)^{1+2} \times \det \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} = -1 \times (2 \times 5 - 4 \times 2) = -1 \times (10 - 8) = -2$
- $C_{13} = (-1)^{1+3} \times \det \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix} = 1 \times (2 \times 0 - 3 \times 2) = -6$
- $C_{21} = (-1)^{2+1} \times \det \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} = -1 \times (2 \times 5 - 3 \times 0) = -10$
- $C_{22} = (-1)^{2+2} \times \det \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = 1 \times (1 \times 5 - 3 \times 2) = -1$
- $C_{23} = (-1)^{2+3} \times \det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = -1 \times (1 \times 0 - 2 \times 2) = 4$
- $C_{31} = (-1)^{3+1} \times \det \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = 1 \times (2 \times 4 - 3 \times 3) = -1$
- $C_{32} = (-1)^{3+2} \times \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = -1 \times (1 \times 4 - 3 \times 2) = 2$
- $C_{33} = (-1)^{3+3} \times \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 1 \times (1 \times 3 - 2 \times 2) = -1$

Thus,

$$B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

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Step 2: Transpose B to get $\text{adj}(A)$:

$$\text{adj}(A) = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$$

Singular and Non-Singular Matrices

A square matrix A is called **singular** if its determinant is zero, i.e., $|A| = 0$. Otherwise, it is **non-singular**.

Example of singular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}$$

Calculate $|A|$:

$$|A| = 1(15 - 12) - 2(12 - 12) + 3(4 - 5) = 3 - 0 - 3 = 0$$

Since $|A| = 0$, A is singular.

Example of non-singular matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Calculate $|A|$:

$$|A| = 0(0-1) - 1(0-1) + 1(1-0) = 0 + 1 + 1 = 2 \neq 0$$

Since $|A| \neq 0$, A is non-singular.

Summary

- Determinant is a scalar value associated with a square matrix.
- Minors and cofactors are used to compute determinants and adjoints.
- Adjoint matrix is the transpose of the cofactor matrix.
- Singular matrices have zero determinant and are not invertible.
- Non-singular matrices have non-zero determinant and are invertible.

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Practice Set

- **Level 1 – Easy**
  - Find the determinant of $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$.
  - Find the minor of element a_{12} in $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.
  - Find the cofactor of element a_{21} in the above matrix.
- **Level 2 – Moderate**
  - Find the adjoint of $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.
  - Determine if the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ is singular or non-singular.
  - Calculate the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.
- **Level 3 – Challenging**
  - Find the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.
  - Prove that $A \cdot \text{adj}(A) = |A|I$ for $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$.
  - Find the determinant and adjoint of $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$.

Answer Key

- **Level 1**
  - Determinant = $3 \times 5 - 4 \times 2 = 15 - 8 = 7$
  - Minor of a_{12} is determinant of $\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = 4 \times 9 - 6 \times 7 = 36 - 42 = -6$
  - Cofactor of a_{21} is $(-1)^{2+1} \times$ minor of a_{21}. Minor is determinant of $\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} = 2 \times 9 - 3 \times 8 = 18 - 24 = -6$. Cofactor = $-1 \times -6 = 6$
- **Level 2**
  - Adjoint of $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ is $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$
  - Determinant of given matrix is $1(1 \times 0 - 4 \times 6) - 2(0 \times 0 - 4 \times 5) + 3(0 \times 6 - 1 \times 5) = 1(0 - 24) - 2(0 - 20) + 3(0 - 5) = -24 + 40 - 15 = 1 \neq 0$, so non-singular.
  - Determinant = $1(3 \times 0 - 1 \times 2) - 0(-1 \times 0 - 1 \times 3) + 2(-1 \times 1 - 3 \times 3) = 1(0 - 2) - 0(0 - 3) + 2(-1 - 9) = -2 + 0 - 20 = -22$
- **Level 3**
  - Adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ is $\begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$
  - For $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $|A| = 1 \times 5 - 3 \times 2 = 5 - 6 = -1$. $\text{adj}(A) = \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$. Then $A \cdot \text{adj}(A) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (1)(5) + (3)(-2) & (1)(-3) + (3)(1) \\ (2)(5) + (5)(-2) & (2)(-3) + (5)(1) \end{bmatrix} = \begin{bmatrix} 5 - 6 & -3 + 3 \\ 10 - 10 & -6 + 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = |A|I$
  - Determinant of $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$ is $2(3 \times 1 - 4 \times 2) - (-1)(1 \times 1 - 4 \times 0) + 0 = 2(3 - 8) + 1(1 - 0) + 0 = 2(-5) + 1 = -10 + 1 = -9$. Adjoint can be calculated similarly.

Quick Reference

Term	Definition
Determinant	Scalar value associated with a square matrix
Minor M_{ij}	Determinant of matrix after deleting i^{th} row and j^{th} column
Cofactor C_{ij}	$(-1)^{i+j} \times M_{ij}$
Adjoint $\text{adj}(A)$	Transpose of cofactor matrix
Singular Matrix	Matrix with zero determinant, not invertible
Non-Singular Matrix	Matrix with non-zero determinant, invertible

Glossary

- **Square Matrix:** A matrix with equal number of rows and columns.
- **Determinant:** A scalar value computed from a square matrix.
- **Minor:** Determinant of submatrix formed by deleting one row and one column.
- **Cofactor:** Signed minor used in determinant expansion.
- **Adjoint:** Transpose of the cofactor matrix.
- **Singular Matrix:** Matrix with zero determinant.
- **Non-Singular Matrix:** Matrix with non-zero determinant.

Inverse of a Matrix by Determinant Method

The inverse of a square matrix A exists only if A is non-singular, i.e., $|A| \neq 0$.

The inverse is given by

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Algorithm to find A^{-1}

1. Calculate $|A|$.
2. If $|A| = 0$, then A is singular and not invertible.
3. If $|A| \neq 0$, calculate the cofactor matrix of A.
4. Find the adjoint $\text{adj}(A)$ by transposing the cofactor matrix.
5. Calculate $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

Worked Example

Find the inverse of $A = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Step 1: Calculate determinant

$$|A| = (-3)(-3) - (2)(5) = 9 - 10 = -1$$

Step 2: Calculate adjoint

$$\text{adj}(A) = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}^T = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}$$

Step 3: Calculate inverse

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$$

Practice Set

- **Level 1 – Easy**
  - Find the inverse of $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$.
  - Calculate the determinant and inverse of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- **Level 2 – Moderate**
  - Find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

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- Verify that $A \cdot A^{-1} = I$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

- **Level 3 – Challenging**
  - Find the inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ using determinant method.
  - Prove that $A \cdot \text{adj}(A) = |A| I$ for a 3x3 matrix A.

Answer Key

- **Level 1**
  - Inverse of $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$ is $\frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$.
  - Determinant of identity matrix is 1; inverse is the identity matrix itself.
- **Level 2**
  - Inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ is $\frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$.
  - Multiplying A and A^{-1} yields the identity matrix.
- **Level 3**
  - Inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ can be found by calculating $|A|$, cofactors, adjoint, and then applying the formula.
  - Proof involves matrix multiplication and properties of determinants.

Quick Reference

Step	Action
1	Calculate $\|A\|$
2	If $\|A\| = 0$, no inverse exists
3	Calculate cofactor matrix
4	Find adjoint by transposing cofactor matrix
5	Calculate inverse $A^{-1} = \frac{1}{\|A\|} \text{adj}(A)$

Glossary

- **Inverse Matrix:** Matrix A^{-1} such that $AA^{-1} = I$.
- **Identity Matrix:** Square matrix with 1's on the diagonal and 0's elsewhere.
- **Singular Matrix:** Matrix with zero determinant, no inverse.
- **Non-Singular Matrix:** Matrix with non-zero determinant, inverse exists.

Area of Triangle Using Determinants

The area Δ of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by

$$\Delta = \frac{1}{2} \left| \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right|$$

Since area is positive, take the absolute value.

If $\Delta = 0$, the points are collinear.

Equation of a Line Using Determinants

The equation of the line passing through points (x_1, y_1) and (x_2, y_2) is

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

Expanding this determinant gives the line equation.

Worked Example

Find the area of the triangle with vertices $(3, 8), (-4, 2), (5, 1)$.

Step 1: Write the determinant

$$\Delta = \frac{1}{2} \left| \det \begin{bmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix} \right|$$

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Step 2: Calculate determinant

$$= \frac{1}{2} |3(2 \times 1 - 1 \times 1) - 8(-4 \times 1 - 1 \times 5) + 1(-4 \times 1 - 2 \times 5)|$$

$$= \frac{1}{2} |3(2 - 1) - 8(-4 - 5) + 1(-4 - 10)| = \frac{1}{2} |3(1) - 8(-9) + 1(-14)|$$

$$= \frac{1}{2} |3 + 72 - 14| = \frac{1}{2} |61| = 30.5$$

Practice Set

- **Level 1 – Easy**
  - Find the area of triangle with vertices $(0, 0), (4, 0), (0, 3)$.
  - Check if points $(1, 2), (3, 4), (5, 6)$ are collinear.
- **Level 2 – Moderate**
  - Find the equation of the line passing through $(2, 3)$ and $(4, 7)$ using determinants.
  - Find the area of triangle with vertices $(1, 1), (4, 5), (7, 2)$.

- **Level 3 – Challenging**
  - Given points $A(1, 3), B(0, 0), D(k, 0)$, find k such that area of triangle ABD is 3.
  - Prove that three points are collinear if the determinant of their coordinate matrix is zero.

Answer Key

- **Level 1**
  - Area = $\frac{1}{2} \times 4 \times 3 = 6$
  - Points are collinear since determinant is zero.
- **Level 2**
  - Equation of line: $y = 2x - 1$
  - Area = 9
- **Level 3**
  - $k = \pm 2$
  - Proof follows from determinant properties and area formula.

Quick Reference

Formula	Description
$\Delta = \frac{1}{2} \mid \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \mid$	Area of triangle
$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$	Equation of line through two points

Glossary

- **Collinear Points:** Points lying on the same straight line.
- **Determinant:** Scalar value used to calculate area and check collinearity.
- **Triangle Area:** Half the absolute value of the determinant of coordinate matrix.