

- Continuity
- Differentiability

## Continuity

**Concept Explanation:** Continuity of a function at a point means the function has no breaks, jumps, or holes at that point. Formally, a function  $f(x)$  is continuous at  $x = m$  if the left-hand limit (LHL) and right-hand limit (RHL) at  $m$  exist and are equal, and the function's value at  $m$  equals this limit:

$$\lim_{x \rightarrow m^-} f(x) = \lim_{x \rightarrow m^+} f(x) = f(m)$$

Indeterminate forms such as  $\frac{0}{0}$ ,  $\frac{\infty}{\infty}$ ,  $0 \times \infty$ ,  $\infty - \infty$ ,  $1^\infty$ ,  $0^0$ , and  $\infty^0$  require further analysis to evaluate limits.

## Formula Derivation for Limits

Key limit formulas used in continuity and calculus include:

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

These formulas simplify evaluating limits near zero or other points.

## Worked Illustration

Find the value of  $k$  such that the function

$$f(x) = \begin{cases} \frac{\sin x - \cos x}{4x - \pi}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$

is continuous at  $x = \frac{\pi}{4}$ .

**Solution:**

Continuity requires:

$$\lim_{x \rightarrow \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right) = k$$

Calculate the limit:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{4x - \pi}$$

Rewrite numerator using sine difference identity:

$$\sin x - \cos x = \sqrt{2} \sin \left( x - \frac{\pi}{4} \right)$$

Rewrite denominator:

$$4x - \pi = 4 \left( x - \frac{\pi}{4} \right)$$

So limit becomes:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} \sin \left( x - \frac{\pi}{4} \right)}{4 \left( x - \frac{\pi}{4} \right)} = \frac{\sqrt{2}}{4} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{\sqrt{2}}{4} \times 1 = \frac{\sqrt{2}}{4}$$

Therefore,  $k = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$ .

## Practice Set

- **Level 1 – Easy:** Verify continuity of  $f(x) = x^2$  at  $x = 2$ .
- **Level 2 – Moderate:** Find  $k$  such that  $f(x) = \begin{cases} x^2 + k, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$  is continuous at  $x = 1$ .
- **Level 3 – Challenging:** Determine if  $f(x) = \frac{\sin x}{x}$  is continuous at  $x = 0$  by defining  $f(0)$  appropriately.

## Answer Key

- Level 1:  $\lim_{x \rightarrow 2} x^2 = 4 = f(2)$ , so continuous.
- Level 2:  $\lim_{x \rightarrow 1^-} f(x) = 1 + k$ ,  $\lim_{x \rightarrow 1^+} f(x) = 3$ , set equal:  $1 + k = 3 \Rightarrow k = 2$ .
- Level 3: Define  $f(0) = 1$  since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , making  $f$  continuous at 0.

## Quick Reference

Concept	Condition
Continuity at $x = m$	$\lim_{x \rightarrow m^-} f(x) = \lim_{x \rightarrow m^+} f(x) = f(m)$
Indeterminate Forms	$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 1^\infty, 0^0, \infty^0$

## Glossary

- **Limit:** The value a function approaches as the input approaches a point.
- **Left-hand limit (LHL):** Limit as  $x$  approaches from the left.
- **Right-hand limit (RHL):** Limit as  $x$  approaches from the right.
- **Continuous function:** Function with no breaks or jumps at a point.
- **Indeterminate form:** Expression where limit evaluation is not straightforward.

## Differentiability

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**Concept Explanation:** Differentiability at a point means the function has a defined derivative (slope) at that point. The derivative represents the instantaneous rate of change of the function.

The derivative of  $f(x)$  at  $x = a$  is defined as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Left-hand derivative (LHD) and right-hand derivative (RHD) are defined as:

$$f'_-(a) = \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}, \quad f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

Function is differentiable at  $a$  if  $f'_-(a) = f'_+(a)$ .

### Relation between Continuity and Differentiability:

- If a function is differentiable at a point, it is continuous there.
- If a function is not continuous at a point, it is not differentiable there.
- A function can be continuous but not differentiable at a point (e.g., sharp corners).

### Formula Derivation for Standard Derivatives

Some standard derivatives are:

Function	Derivative
$x^n$	$nx^{n-1}$
$k$ (constant)	0
$e^x$	$e^x$
$\log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

## Rules of Derivatives

Product Rule:

$$\frac{d}{dx}[uv] = u \frac{dv}{dx} + v \frac{du}{dx}$$

Quotient Rule:

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

### Worked Illustration

Differentiate  $y = \tan^{-1} \left( \frac{\cos x - \sin x}{\cos x + \sin x} \right)$ .

Solution:

$$\text{Let } y = \tan^{-1} \left( \frac{\cos x - \sin x}{\cos x + \sin x} \right).$$

Divide numerator and denominator inside the argument by  $\cos x$ :

$$\frac{1 - \tan x}{1 + \tan x} = \tan \left( \frac{\pi}{4} - x \right)$$

$$\text{So, } y = \tan^{-1} \left[ \tan \left( \frac{\pi}{4} - x \right) \right] = \frac{\pi}{4} - x.$$

Differentiate:

$$\frac{dy}{dx} = 0 - 1 = -1$$

## Practice Set

- **Level 1 – Easy:** Find  $\frac{d}{dx}(x^3)$ .
- **Level 2 – Moderate:** Differentiate  $y = \sin x \cos x$  using product rule.
- **Level 3 – Challenging:** Find  $\frac{dy}{dx}$  if  $y = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$ .

## Answer Key

- Level 1:  $\frac{d}{dx}(x^3) = 3x^2$ .
- Level 2: Using product rule,  $\frac{dy}{dx} = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$ .
- Level 3: Simplify inside inverse tangent as  $\tan\left(\frac{\pi}{4} - x\right)$ , so derivative is  $-1$ .

## Quick Reference

Function	Derivative
$x^n$	$nx^{n-1}$
$e^x$	$e^x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
Product Rule	$(uv)' = u'v + uv'$
Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

## Glossary

- **Derivative:** Instantaneous rate of change of a function.
- **Left-hand derivative (LHD):** Derivative approaching from the left.
- **Right-hand derivative (RHD):** Derivative approaching from the right.
- **Differentiable function:** Function with equal LHD and RHD at a point.
- **Product rule:** Rule to differentiate product of two functions.
- **Quotient rule:** Rule to differentiate quotient of two functions.

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