Indefinite Integral

Concept Explanation: The indefinite integral of a function $f(x)$ is the family of all antiderivatives $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$. It is denoted by $\int f(x)\,dx = F(x) + C$, where C is the constant of integration. Integration is the inverse process of differentiation.

Methods of Integration

- **Integration by Substitution:** Change the variable of integration from x to t where $x = g(t)$. Then,

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt.$$

- **Integration by Partial Fractions:** For rational functions $\frac{f(x)}{g(x)}$, if degree of numerator $\geq$ degree of denominator, perform polynomial division first. Then decompose into partial fractions based on the factorization of denominator and integrate each simpler fraction.
- **Integration by Parts:** For functions $U(x)$ and $V(x)$,

$$\int U\,dV = UV - \int V\,dU.$$

Formula Derivation

For Integration by Parts, starting from the product rule:

$$\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx} \implies U\frac{dV}{dx} = \frac{d}{dx}(UV) - V\frac{dU}{dx}.$$

Integrating both sides,

$$\int U\,dV = UV - \int V\,dU.$$

Worked Illustrations

Example 1: Integrate $\int \tan^4(\sqrt{x})\sec^2(\sqrt{x})/\sqrt{x}\,dx$.

Solution: Let $t = \sqrt{x}$, then $dt = \frac{1}{2\sqrt{x}}dx \Rightarrow dx = 2t\,dt$.

Rewrite integral:

$$I = \int \tan^4(t)\sec^2(t)\frac{1}{t} \times 2t\,dt = 2\int \tan^4(t)\sec^2(t)\,dt.$$

Substitute $u = \tan t \Rightarrow du = \sec^2 t\,dt$, so

$$I = 2\int u^4\,du = 2\frac{u^5}{5} + C = \frac{2}{5}\tan^5(\sqrt{x}) + C.$$

Example 2: Find $\int \frac{dx}{3x^2+13x-10}$.

Solution: Factor denominator or complete the square:

$$3x^2 + 13x - 10 = 3\left(x^2 + \frac{13}{3}x - \frac{10}{3}\right).$$

Complete the square:

$$x^2 + \frac{13}{3}x = \left(x + \frac{13}{6}\right)^2 - \left(\frac{13}{6}\right)^2.$$

Rewrite integral:

$$I = \frac{1}{3}\int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}.$$

Put $t = x + \frac{13}{6}$, $dt = dx$, then

$$I = \frac{1}{3}\int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2} = \frac{1}{3}\int \frac{dt}{(t - \frac{17}{6})(t + \frac{17}{6})}.$$

Use partial fractions:

$$\frac{1}{(t-a)(t+a)} = \frac{A}{t-a} + \frac{B}{t+a}.$$

Solving for A, B, integrate and simplify to get

$$I = \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right| + C.$$

Practice Set

- **Level 1 – Easy:**
    - Find $\int x^3\,dx$.
    - Evaluate $\int \sin x\,dx$.
    - Compute $\int e^{2x}\,dx$.
- **Level 2 – Moderate:**

- Evaluate $\int xe^x\,dx$ using integration by parts.
- Find $\int \frac{dx}{x^2-4}$ using partial fractions.
- Calculate $\int \tan^3 x\,dx$ using substitution.

- **Level 3 – Challenging:**
  - Evaluate $\int \frac{x^2+3x+2}{(x+1)(x+2)^2}\,dx$ by partial fractions.
  - Find $\int x^2 \sin x\,dx$ using integration by parts twice.
  - Compute $\int \frac{dx}{x^3+x}$ by partial fraction decomposition.

Answer Key

- $\int x^3\,dx = \frac{x^4}{4} + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$
- $\int xe^x\,dx = xe^x - e^x + C$
- $\int \frac{dx}{x^2-4} = \frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + C$
- $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \log|\cos x| + C$
- $\int \frac{x^2+3x+2}{(x+1)(x+2)^2}\,dx = -\frac{1}{x+2} + \log|x+1| + C$
- $\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$
- $\int \frac{dx}{x^3+x} = \frac{1}{2}\log|x| - \frac{1}{2}\log|x^2+1| + C$

Quick Reference

Integral	Result
$\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + C, n \neq -1$
$\int e^{ax}\,dx$	$\frac{e^{ax}}{a} + C$
$\int \sin(ax)\,dx$	$-\frac{1}{a}\cos(ax) + C$
$\int \cos(ax)\,dx$	$\frac{1}{a}\sin(ax) + C$
Integration by Parts	$\int U\,dV = UV - \int V\,dU$
Integration by Substitution	$\int f(g(t))g'(t)\,dt$

Glossary

- **Integral:** The antiderivative or primitive of a function.
- **Indefinite Integral:** Integral without limits, representing a family of functions.
- **Integration by Substitution:** Method changing variable to simplify integral.
- **Partial Fractions:** Decomposition of rational functions into simpler fractions.
- **Integration by Parts:** Technique based on product rule for differentiation.
- **Constant of Integration (C):** Arbitrary constant added to indefinite integrals.

Definite Integral

Concept Explanation: The definite integral of a function $f(x)$ from a to b is the numeric value

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$. It represents the net area under the curve $y = f(x)$ between $x = a$ and $x = b$.

Properties of Definite Integrals

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is constant
- Splitting integral: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- Symmetry: For even function $f(-x) = f(x)$, $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
- For odd function $f(-x) = -f(x)$, $\int_{-a}^a f(x)\,dx = 0$

Worked Illustrations

Example 3: Evaluate $I = \int_0^{\pi/4} \sin^3(2t)\cos(2t)\,dt$.

Solution: Let $u = \sin(2t)$, then $du = 2\cos(2t)dt \Rightarrow \cos(2t)dt = \frac{1}{2}du$.

Change limits: when $t = 0, u = 0$; when $t = \pi/4, u = 1$.

Rewrite integral:

$$I = \int_0^{\pi/4} \sin^3(2t)\cos(2t)\,dt = \frac{1}{2}\int_0^1 u^3\,du = \frac{1}{2} \times \frac{u^4}{4}\Big|_0^1 = \frac{1}{8}.$$

Example 4: Evaluate $\int_1^2 \frac{x}{(x+1)(x+2)}\,dx$.

Solution: Decompose into partial fractions:

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Multiply both sides by $(x+1)(x+2)$:

$$x = A(x+2) + B(x+1) = (A+B)x + (2A+B).$$

Equate coefficients:

$$A + B = 1, \quad 2A + B = 0.$$

Solving gives $A = -1, B = 2$.

Rewrite integral:

$$I = \int_1^2 \left(\frac{-1}{x+1} + \frac{2}{x+2}\right)dx = -\int_1^2 \frac{dx}{x+1} + 2\int_1^2 \frac{dx}{x+2}.$$

Integrate:

$$I = -[\log|x+1|]_1^2 + 2[\log|x+2|]_1^2 = -\log 3 + \log 2 + 2\log 4 - 2\log 3 = \log\frac{32}{27}.$$

Practice Set

- **Level 1 – Easy:**
  - Evaluate $\int_0^1 x^2\,dx$.
  - Find $\int_1^2 \frac{1}{x}\,dx$.
  - Calculate $\int_0^\pi \sin x\,dx$.
- **Level 2 – Moderate:**
  - Evaluate $\int_0^1 xe^{x^2}\,dx$ using substitution.
  - Find $\int_1^3 \frac{2x+1}{x^2+x}\,dx$ using partial fractions.
  - Calculate $\int_{-1}^1 x^3\,dx$ and explain the result.
- **Level 3 – Challenging:**
  - Evaluate $\int_0^{\pi/2} \sin^2 x\,dx$ using symmetry and reduction formulas.
  - Find $\int_0^2 \frac{x^2}{x^3+1}\,dx$ using substitution and partial fractions.
  - Prove $\int_{-a}^a f(x)\,dx = 0$ if f is odd.

Answer Key

- $\int_0^1 x^2\,dx = \frac{1}{3}$
- $\int_1^2 \frac{1}{x}\,dx = \log 2$
- $\int_0^\pi \sin x\,dx = 2$
- $\int_0^1 xe^{x^2}\,dx = \frac{e-1}{2}$
- $\int_1^3 \frac{2x+1}{x^2+x}\,dx = 2\log 3 - \log 2$
- $\int_{-1}^1 x^3\,dx = 0$ because x^3 is odd.
- $\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$
- $\int_0^2 \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\log 9$
- Proof: For odd f, $f(-x) = -f(x)$, so

$$\int_{-a}^a f(x)\,dx = 0.$$

Quick Reference

Property	Formula
Fundamental Theorem	$\int_a^b f(x)\,dx = F(b) - F(a)$
Reversing limits	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Linearity	$\int_a^b [kf(x) + g(x)]\,dx = k\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
Splitting integral	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
Even function	$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
Odd function	$\int_{-a}^a f(x)\,dx = 0$

Glossary

- **Definite Integral:** Integral with limits, giving numeric value.
- **Limits of Integration:** The values a and b between which the integral is evaluated.
- **Antiderivative:** A function whose derivative is the integrand.
- **Even Function:** $f(-x) = f(x)$, symmetric about y-axis.
- **Odd Function:** $f(-x) = -f(x)$, symmetric about origin.
- **Linearity:** Property allowing splitting and scaling of integrals.