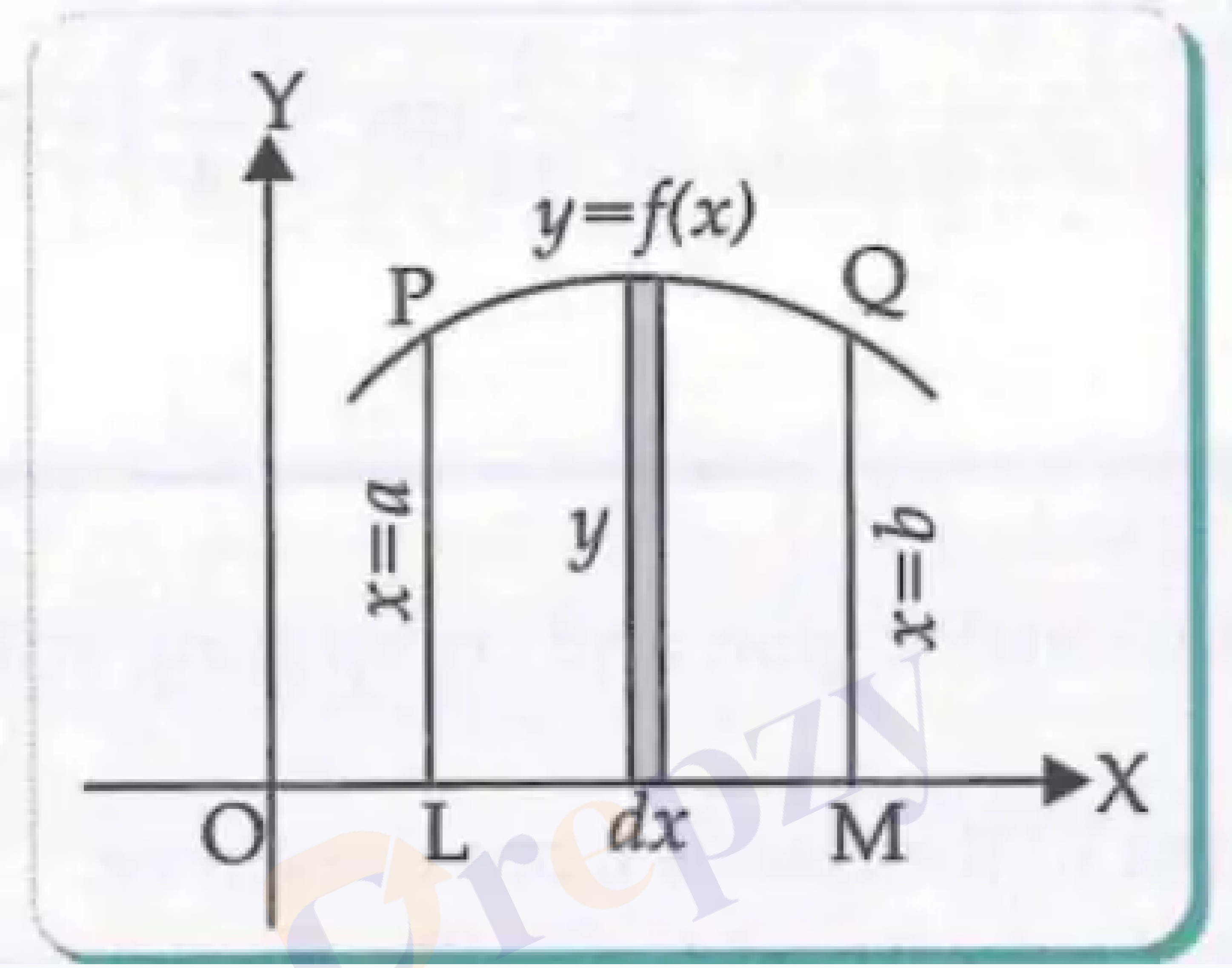
Area Under Simple Curves

The area bounded by a curve and the coordinate axes can be found using definite integrals. Consider a curve defined by $y = f(x)$ between $x = a$ and $x = b$. The area under the curve and above the x-axis is approximated by summing the areas of thin vertical strips of width dx and height $y = f(x)$.

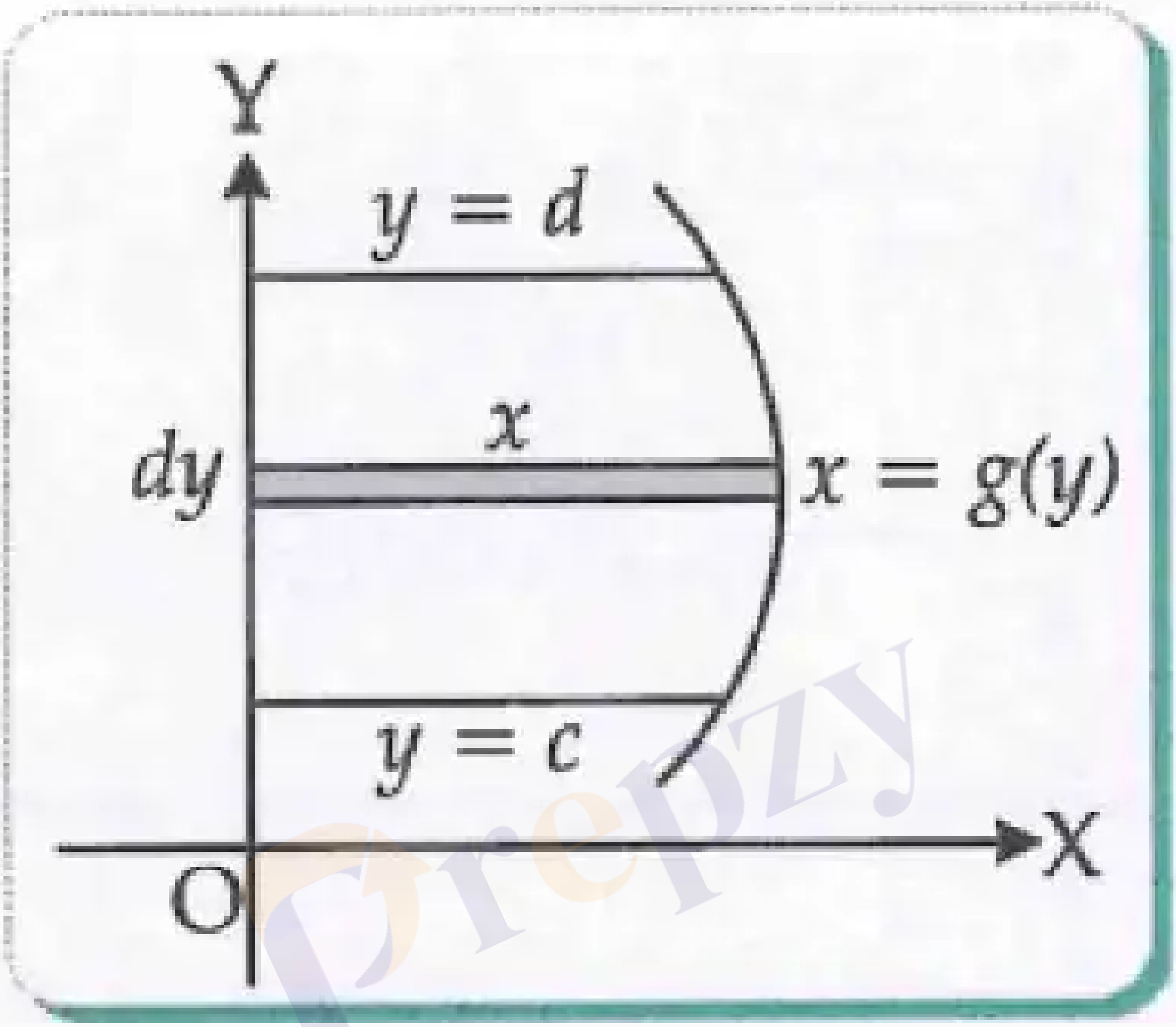
The area of an elementary strip is $dA = y\,dx = f(x)\,dx$. Summing these strips from a to b gives the total area:

$$A = \int_a^b f(x)\,dx$$



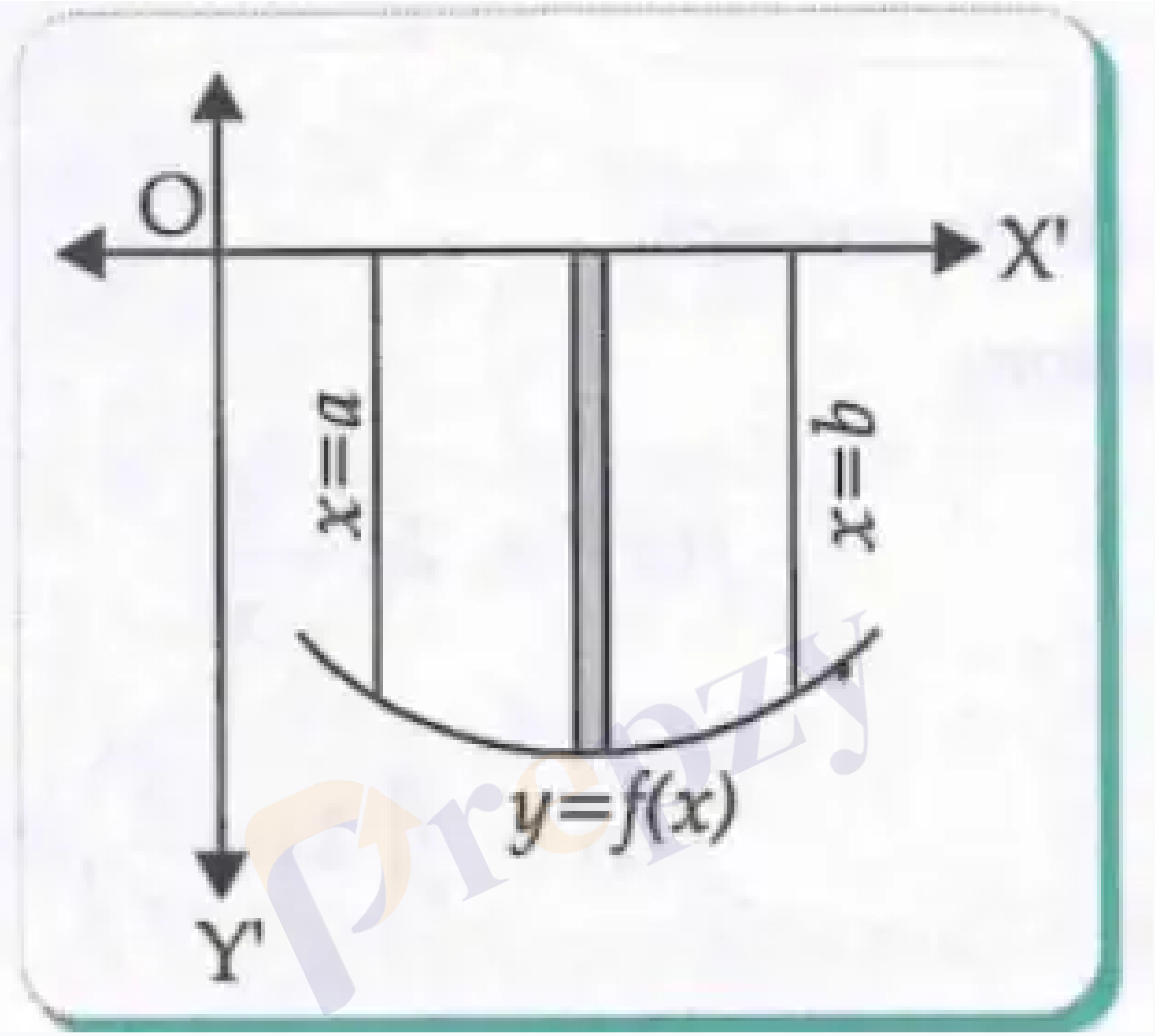
Similarly, if the curve is given by $x = g(y)$ between $y = c$ and $y = d$, the area bounded by the curve, y-axis, and the lines $y = c$ and $y = d$ is:

$$A = \int_c^d g(y)\,dy$$



If the curve lies below the x-axis, i.e., $f(x) < 0$ for $x \in [a, b]$, the definite integral $\int_a^b f(x)\,dx$ is negative. The area bounded by the curve and the x-axis is the absolute value of this integral:

$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$



If the curve crosses the x-axis, the total area bounded by the curve and the x-axis between $x = u$ and $x = b$ is the sum of the absolute values of the areas above and below the x-axis:

$$A = |A_1| + |A_2|$$

Worked Illustration: Area Enclosed by an Ellipse

Find the area enclosed by the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution:

The ellipse is symmetric about both axes. The total area is four times the area in the first quadrant:

$$A = 4 \times \text{Area in first quadrant} = 4\int_0^a y\,dx$$

From the ellipse equation, solve for y:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Substitute into the integral:

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

Use the standard integral formula:

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Evaluate the definite integral:

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

At $x = a$:

$$\frac{a}{2} \times 0 + \frac{a^2}{2} \times \frac{\pi}{2} = \frac{a^2\pi}{4}$$

At $x = 0$:

$$0 + 0 = 0$$

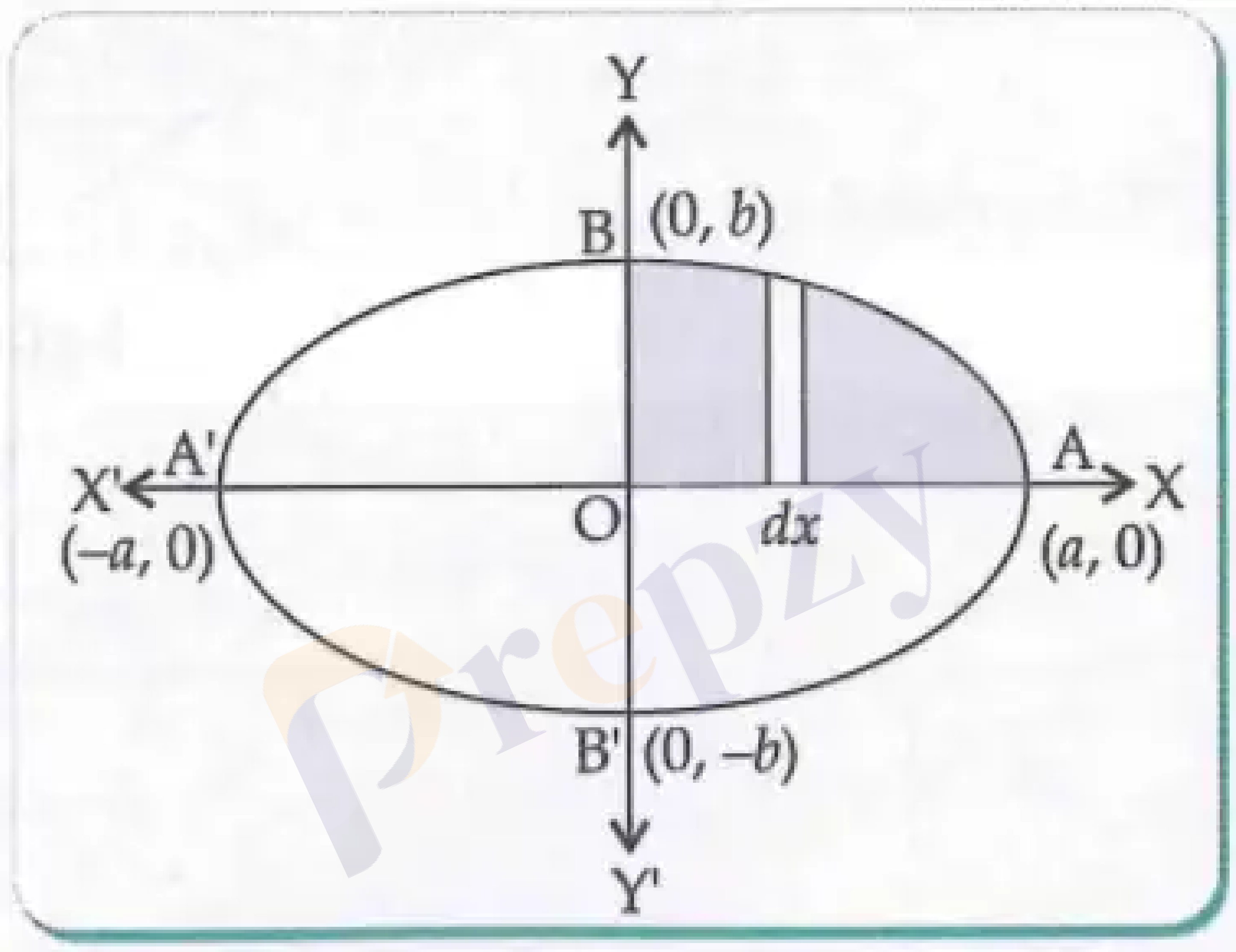
Therefore,

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{a^2\pi}{4}$$

Substitute back:

$$A = \frac{4b}{a} \times \frac{a^2\pi}{4} = \pi ab$$

Thus, the area enclosed by the ellipse is πab square units.



Worked Illustration: Area Bounded by Parabola and Line

Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution:

The parabola is symmetric about the y-axis. The region bounded by $y = x^2$ and $y = 4$ is symmetric about the y-axis. Calculate the area in the first quadrant and multiply by 2:

Express x in terms of y:

$$x = \sqrt{y}$$

The area in the first quadrant is:

$$\int_0^4 \sqrt{y}\, dy$$

Therefore, total area:

$$A = 2\int_0^4 \sqrt{y}\, dy = 2\int_0^4 y^{1/2}\, dy$$

Integrate:

$$\int y^{1/2}\, dy = \frac{2}{3} y^{3/2} + C$$

Evaluate definite integral:

$$\int_0^4 y^{1/2}\, dy = \left[\frac{2}{3} y^{3/2}\right]_0^4 = \frac{2}{3}(4)^{3/2} - 0 = \frac{2}{3} \times 8 = \frac{16}{3}$$

Multiply by 2:

$$A = 2 \times \frac{16}{3} = \frac{32}{3}$$

Thus, the area of the region bounded by $y = x^2$ and $y = 4$ is $\frac{32}{3}$ square units.

Practice Set

- **Level 1 – Easy**
  - Find the area under the curve $y = 3x$ between $x = 0$ and $x = 2$.
  - Calculate the area bounded by $y = x^2$ and the x-axis from $x = 0$ to $x = 3$.
- **Level 2 – Moderate**
  - Find the area bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 1$ and $x = 4$.
  - Calculate the area bounded by the curve $x = y^2$, the y-axis, and the lines $y = 1$ and $y = 3$.
- **Level 3 – Challenging**
  - Find the area enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$.
  - Calculate the area bounded by the curve $y = x^3$ and the line $y = 8$.

Answer Key

- **Level 1**
  - $\int_0^2 3x\,dx = \left[\frac{3x^2}{2}\right]_0^2 = \frac{3\times4}{2} = 6$
  - $\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} = 9$
- **Level 2**
  - $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$
  - $\int_1^3 y^2\,dy = \left[\frac{y^3}{3}\right]_1^3 = \frac{27-1}{3} = \frac{26}{3}$
- **Level 3**
  - Area of ellipse $= \pi ab = \pi \times 3 \times 4 = 12\pi$

  - Set $y = x^3$, line $y = 8$ implies $x = 2$. Area:

$$\int_0^2 (8 - x^3)\,dx = \left[8x - \frac{x^4}{4}\right]_0^2 = (16 - 4) - 0 = 12$$

Quick Reference

Concept	Formula
Area under $y = f(x)$ from a to b	$\int_a^b f(x)\,dx$
Area under $x = g(y)$ from c to d	$\int_c^d g(y)\,dy$
Area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	πab

Glossary

- **Arbitrary:** A constant or value that can be chosen freely within a problem.
- **Curve:** A continuous and smooth flowing line without sharp angles.
- **Definite Integral:** The integral of a function over a specific interval, representing area under the curve.
- **Ordinates:** The y-coordinates of points on a curve.
- **Abscissa:** The x-coordinates of points on a curve.