Conditional Probability and Multiplication Theorem on Probability

Concept Explanation: Probability measures the likelihood of an event occurring. For an event E in sample space S, probability is $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ is the number of favorable outcomes and $n(S)$ is the total number of outcomes.

Events can be:

- **Mutually Exclusive:** Events that cannot occur simultaneously, i.e., $A \cap B = \emptyset$.
- **Independent:** Occurrence of one event does not affect the other.
- **Exhaustive:** Events whose union covers the entire sample space and are pairwise disjoint.

Conditional Probability is the probability of event A given event B has occurred, denoted $P(A|B) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) \neq 0$.

Formula Derivation

By definition, the conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplying both sides by $P(B)$, we get the multiplication theorem:

$$P(A \cap B) = P(B) \times P(A|B)$$

Similarly, $P(B \cap A) = P(A) \times P(B|A)$.

Worked Illustrations and Solved Examples

Example 1: A family has two children. Find the probability both are boys given at least one is a boy.

Solution:

Sample space $S = \{(b,b),(b,g),(g,b),(g,g)\}$, $n(S) = 4$.

Event E: both children are boys $= \{(b,b)\}$, $n(E) = 1$.

Event F: at least one boy $= \{(b,b),(b,g),(g,b)\}$, $n(F) = 3$.

$E \cap F = \{(b,b)\}$, $n(E \cap F) = 1$.

Calculate probabilities:

$$P(F) = \frac{3}{4}, \quad P(E \cap F) = \frac{1}{4}$$

Conditional probability:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 2: Ten cards numbered 1 to 10 are mixed. If a card drawn is known to be greater than 3, find the probability it is even.

Solution:

Sample space $S = \{1,2,3,4,5,6,7,8,9,10\}$, $n(S) = 10$.

Event A: card is even $= \{2,4,6,8,10\}$, $n(A) = 5$.

Event B: card is greater than 3 $= \{4,5,6,7,8,9,10\}$, $n(B) = 7$.

$A \cap B = \{4, 6, 8, 10\}, n(A \cap B) = 4.$

Calculate probabilities:

$$P(A) = \frac{5}{10} = \frac{1}{2}, \quad P(B) = \frac{7}{10}, \quad P(A \cap B) = \frac{4}{10} = \frac{2}{5}$$

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{5}}{\frac{7}{10}} = \frac{2}{5} \times \frac{10}{7} = \frac{4}{7}$$

Practice Set

- **Level 1 – Easy:** A die is rolled. Find the probability of getting a 4 given the number is even.
- **Level 2 – Moderate:** Two cards are drawn without replacement from a deck of 52. Find the probability both are kings.
- **Level 3 – Challenging:** In a box, there are 5 red and 7 blue balls. Two balls are drawn one after another without replacement. Find the probability both are red.

Answer Key

- Level 1: $P(4|\text{even}) = \frac{P(4)}{P(2,4,6)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$
- Level 2: $P(\text{both kings}) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$
- Level 3: $P(\text{both red}) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$

Quick Reference

Formula	Description
$P(A\|B) = \frac{P(A \cap B)}{P(B)}$	Conditional probability of A given B
$P(A \cap B) = P(B) \times P(A\|B)$	Multiplication theorem
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$	Probability of union of two events

Glossary

- **Sample Space (S):** Set of all possible outcomes.
- **Event (E):** Subset of sample space.
- **Mutually Exclusive:** Events that cannot occur together.
- **Independent Events:** Events where occurrence of one does not affect the other.
- **Conditional Probability:** Probability of an event given another event has occurred.

Bayes Theorem

Concept Explanation: Bayes theorem provides a way to update probabilities based on new information. If $E_1, E_2, \ldots, E_n$ form a partition of sample space S and A is an event with $P(A) > 0$, then the probability of E_i given A is:

Formula Derivation

By definition of conditional probability:

$$P(E_i|A) = \frac{P(E_i \cap A)}{P(A)}$$

Since E_i and A occur together, and E_i are disjoint partitions,

$$P(E_i \cap A) = P(E_i) \times P(A|E_i)$$

Also, total probability of A is:

$$P(A) = \sum_{j=1}^{n} P(E_j)P(A|E_j)$$

Therefore, Bayes theorem is:

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

Worked Illustrations and Solved Examples

Example 3: A construction job has a 0.65 probability of strike. Probability of completion on time is 0.80 if no strike, 0.32 if strike. Find probability job completes on time.

Solution:

Let F: strike, E: job completes on time.

$$P(F) = 0.65, \quad P(\overline{F}) = 0.35$$

$$P(E|F) = 0.32, \quad P(E|\overline{F}) = 0.80$$

By total probability theorem:

$$P(E) = P(F)P(E|F) + P(\overline{F})P(E|\overline{F}) = 0.65 \times 0.32 + 0.35 \times 0.80 = 0.208 + 0.28 = 0.488$$

Example 4: Three boxes I, II, III contain coins: I has 2 gold, II has 2 silver, III has 1 gold and 1 silver. A box is chosen at random and a coin drawn. If the coin is gold, find probability the other coin in the box is gold.

Solution:

Let A, B, C be choosing boxes I, II, III respectively.

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

Event E: coin drawn is gold.

$$P(E|A) = 1, \quad P(E|B) = 0, \quad P(E|C) = \frac{1}{2}$$

By Bayes theorem:

$$P(A|E) = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B) + P(C)P(E|C)} = \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3} + 0 + \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Practice Set

- **Level 1 – Easy:** A box contains 3 red and 2 blue balls. One ball is drawn. If it is red, find the probability it came from the first box if two boxes have different compositions.
- **Level 2 – Moderate:** A factory has two machines producing items. Machine A produces 60% items with 2% defect rate, Machine B produces 40% items with 3% defect rate. If an item is defective, find probability it was produced by Machine A.
- **Level 3 – Challenging:** Three boxes contain different numbers of defective and non-defective items. Given probabilities of choosing each box and drawing a defective item, find the probability the defective item came from a specific box.

Answer Key

- Level 1: Use Bayes theorem with given box probabilities and red ball probabilities.
- Level 2: $P(A|D) = \frac{P(A)P(D|A)}{P(A)P(D|A)+P(B)P(D|B)} = \frac{0.6\times0.02}{0.6\times0.02+0.4\times0.03} = \frac{0.012}{0.012+0.012} = \frac{1}{2}$
- Level 3: Apply Bayes theorem with given data.

Quick Reference

Formula	Description
$P(E_i\|A) = \frac{P(E_i)P(A\|E_i)}{\sum_{j=1}^{n} P(E_j)P(A\|E_j)}$	Bayes theorem for event E_i given A
$P(A) = \sum_{j=1}^{n} P(E_j)P(A\|E_j)$	Total probability of A

Glossary

- **Prior Probability:** Probability of an event before new evidence.
- **Posterior Probability:** Updated probability after considering new evidence.
- **Partition:** A set of mutually exclusive and exhaustive events.

Random Variable and its Probability Distributions

Concept Explanation: A random variable X is a real-valued function defined on the sample space of an experiment. It assigns a numerical value to each outcome.

Probability Distribution: The set of values X can take along with their probabilities is called the probability distribution.

Types:

- **Discrete Random Variable:** Takes finite or countably infinite values.
- **Continuous Random Variable:** Takes any value in an interval.

Formula Derivation

Mean or Expectation μ of X:

$$\mu = E(X) = \sum_i x_i P(x_i)$$

where x_i are values of X and $P(x_i)$ their probabilities.

Worked Illustrations and Solved Examples

Example 5: Two cards drawn with replacement from 52 cards. Find probability distribution of number of aces.

Solution:

Random variable X: number of aces drawn, possible values $0, 1, 2$.

Probability of ace $= \frac{4}{52} = \frac{1}{13}$, non-ace $= \frac{48}{52} = \frac{12}{13}$.

Calculate probabilities:

$$P(X = 0) = P(\text{no ace in both draws}) = \left(\frac{12}{13}\right)^2 = \frac{144}{169}$$

$$P(X = 1) = P(\text{one ace}) = 2 \times \frac{1}{13} \times \frac{12}{13} = \frac{24}{169}$$

$$P(X = 2) = P(\text{two aces}) = \left(\frac{1}{13}\right)^2 = \frac{1}{169}$$

Probability distribution table:

X	0	1	2
$P(X)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Practice Set

- **Level 1 – Easy:** Toss a coin 3 times. Find probability distribution of number of heads.
- **Level 2 – Moderate:** A die is rolled twice. Find probability distribution of sum of numbers.
- **Level 3 – Challenging:** Given a probability distribution with unknown constant k, find k and calculate probabilities for given events.

Answer Key

- Level 1: $P(X = k) = \binom{3}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{3-k}$ for $k = 0, 1, 2, 3$.
- Level 2: Calculate sums and their probabilities by counting outcomes.
- Level 3: Use $\sum P(X = x_i) = 1$ to find k, then compute required probabilities.

Quick Reference

Formula	Description
$E(X) = \sum x_i P(x_i)$	Mean or expectation of random variable
$\sum P(x_i) = 1$	Sum of probabilities equals 1

Glossary

- **Random Variable:** Function assigning numerical values to outcomes.
- **Probability Distribution:** Mapping of values to their probabilities.
- **Expectation:** Weighted average of values.
- **Discrete Variable:** Takes countable values.
- **Continuous Variable:** Takes values in an interval.