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Centre of Mass and Motion of Rotational Particles

Kinds of Motion of Rigid Body

Rigid bodies can exhibit different types of motion depending on how their particles move:

- **Pure Translational Motion:** All particles of the body move with the same velocity at a given instant. For example, a car moving in a straight line.
- **Pure Rotational Motion:** The body rotates about a fixed axis. Every particle moves in a circle perpendicular to the axis, centered on the axis. For example, a potter's wheel.
- **Combination of Translational and Rotational Motion:** When a rigid body is not fixed, it may have both translational and rotational motion simultaneously. For example, a vehicle's wheel rolling on the road.

Centre of Mass of a Two-Particle System

The position vector of the centre of mass (C.M.) of a system of two particles is given by the weighted average of their position vectors:

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

where m_1, m_2 are the masses and \vec{r}_1, \vec{r}_2 are the position vectors of the particles.

Momentum Conservation

The total linear momentum \vec{p} of a system of particles is the product of the total mass M and the velocity of the centre of mass \vec{v} :

$$\vec{p} = M\vec{v} = \sum_{i=1}^n m_i \vec{v}_i$$

Differentiating with respect to time gives:

$$\frac{d\vec{p}}{dt} = M \frac{d\vec{v}}{dt} = M\vec{a} = \vec{F}_{\text{ext}}$$

This is Newton's Second Law for a system of particles, where \vec{F}_{ext} is the net external force.

For an isolated system where $\vec{F}_{\text{ext}} = 0$, momentum is conserved:

$$\frac{d\vec{p}}{dt} = 0 \implies \vec{p} = \text{constant} \implies M\vec{v} = \text{constant}$$

Moment of Force or Torque

Torque $\vec{\tau}$ is the turning effect of a force about an axis and is given by the vector product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Angular Momentum and its Conservation

Angular momentum \vec{L} of a particle about an axis is the moment of its linear momentum about that axis:

$$\vec{L} = \vec{r} \times \vec{p} = rp \sin \phi = d \times p$$

where $d = r \sin \phi$ is the perpendicular distance from the axis to the line of action of \vec{p} . Angular momentum is a vector perpendicular to both \vec{r} and \vec{p} , direction given by the right-hand screw rule.

The rate of change of angular momentum equals the torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

For an isolated system with no external torque:

$$\vec{\tau}_{\text{ext}} = 0 \implies \frac{d\vec{L}}{dt} = 0 \implies \vec{L} = \text{constant}$$

Equilibrium of Rigid Bodies

First Condition (Translational Equilibrium): The net external force on the body is zero, so the body remains at rest or moves with constant velocity:

$$\sum \vec{F}_i = 0$$

Second Condition (Rotational Equilibrium): The net external torque on the body is zero, so the body does not rotate or rotates with constant angular velocity:

$$\sum \vec{\tau}_i = 0$$

Translational static equilibrium can be:

- Stable equilibrium
- Unstable equilibrium
- Neutral equilibrium

Principle of Moments

A body is in rotational equilibrium if the algebraic sum of moments of all forces about a fixed point is zero.

Key Definitions

- **Rigid Body:** A system of particles where distances between particles remain constant under external forces.

- **Centre of Mass:** The point where the entire mass of the body can be considered to be concentrated.
- **Centre of Gravity:** The point where the weight of the body acts and the total gravitational torque is zero.

Key Formulae

Position Vector of Centre of Mass for n-Particle System:

$$\vec{r} = \frac{\sum_{i=1}^n m_i \vec{r}_i}{M} \quad \text{where} \quad M = \sum_{i=1}^n m_i$$

For Two-Particle System:

$$\vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Coordinates of Centre of Mass:

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

Velocity of Centre of Mass:

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

Angular Momentum:

$$\vec{L} = \vec{r} \times m\vec{v}$$

Equations of Rotational Motion:

- Angular displacement: $\theta = \omega_i t + \frac{1}{2} \alpha t^2$
- Linear velocity: $v = r\omega$, where $\omega = 2\pi\nu = \frac{2\pi}{T}$
- Linear acceleration: $a = r\alpha$
- Centripetal acceleration: $a_c = \frac{v^2}{r} = r\omega^2$

Torque:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Work Done by Torque:

$$dW = \tau d\theta$$

Power of Torque:

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

R S Das in a Cinema hall met Chiranjeet mall area and Ram Chandran behind Central door.

- R – Ring
- S – Sphere
- D – Disc
- C – Centre

Ring, sphere, and disc have centres of mass at their respective centres.

- C – Cylinder
- m – mid-point
- a – Axis

Cylinders have centres of mass at the mid-point on their axis.

- R – Rectangular lamina
- c – cube
- c – cross-point
- d – diagonal

Rectangular lamina and cube have centres of mass at the cross point of their diagonals.

Solved Examples

Example 1: Find the position of the centre of mass of two particles of masses 3 kg and 5 kg located at points (2, 0, 0) m and (0, 4, 0) m respectively.

Solution:

Given: $m_1 = 3 \text{ kg}$, $\vec{r}_1 = (2, 0, 0) \text{ m}$

$$m_2 = 5 \text{ kg}, \vec{r}_2 = (0, 4, 0) \text{ m}$$

$$\text{Total mass } M = 3 + 5 = 8 \text{ kg}$$

Position vector of centre of mass:

$$\vec{r} = \frac{3 \times (2, 0, 0) + 5 \times (0, 4, 0)}{8} = \frac{(6, 0, 0) + (0, 20, 0)}{8} = \left(\frac{6}{8}, \frac{20}{8}, 0 \right) = (0.75, 2.5, 0) \text{ m}$$

So, the centre of mass is at (0.75 m, 2.5 m, 0).

Practice Set

Conceptual Questions:

- **Level 1:** Define the centre of mass of a system of particles.
- **Level 2:** Explain the difference between pure translational motion and pure rotational motion of a rigid body.

Application-based Question:

- **Level 3:** Two particles of masses 4 kg and 6 kg are located at points (1, 2, 0) m and (3, 0, 0) m respectively. Calculate the velocity of the centre of mass if the particles have velocities $\vec{v}_1 = (2, 0, 0) \text{ m/s}$ and $\vec{v}_2 = (0, 3, 0) \text{ m/s}$.

Answer Key

Conceptual Questions:

- **Level 1:** The centre of mass is the point in a system of particles where the entire mass can be considered to be concentrated for analysis of translational motion.
- **Level 2:** Pure translational motion means all particles move with the same velocity; pure rotational motion means the body rotates about a fixed axis with particles moving in circles around it.

Application-based Question:

Given:

$$m_1 = 4 \text{ kg}, \vec{v}_1 = (2, 0, 0) \text{ m/s}$$

$$m_2 = 6 \text{ kg}, \vec{v}_2 = (0, 3, 0) \text{ m/s}$$

$$\text{Total mass } M = 10 \text{ kg}$$

Velocity of centre of mass:

$$\vec{v}_{CM} = \frac{4 \times (2, 0, 0) + 6 \times (0, 3, 0)}{10} = \frac{(8, 0, 0) + (0, 18, 0)}{10} = (0.8, 1.8, 0) \text{ m/s}$$

So, the velocity of the centre of mass is (0.8 m/s, 1.8 m/s, 0).

Moment of Inertia and Radius of Gyration

Principle of Conservation of Angular Momentum

When no external torque acts on a system, the total angular momentum remains constant:

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_n = \text{constant}$$

Laws of Rotational Motion

- **First Law:** A body remains at rest or in uniform rotation unless acted upon by an external torque.
- **Second Law:** The rate of change of angular momentum is proportional to the external torque applied.
- **Third Law:** For every torque exerted by body A on body B, body B exerts an equal and opposite torque on body A.

Moment of Inertia

Moment of inertia I about an axis is the measure of a body's resistance to change in rotational motion about that axis. For a particle:

$$I = mr^2$$

where m is mass and r is perpendicular distance from the axis.

Moment of inertia is a scalar with SI unit $\text{kg} \cdot \text{m}^2$.

Radius of Gyration

The radius of gyration K is the distance from the axis at which the entire mass can be considered concentrated to have the same moment of inertia:

$$K = \sqrt{\frac{\sum m_i r_i^2}{M}}$$

Kinetic Energy of Rotation

The rotational kinetic energy of a body is:

$$KE = \frac{1}{2} I \omega^2$$

Key Formulae for Moment of Inertia

- Circular ring (about center, perpendicular to plane): $I = MR^2$
- Circular disc (about center, perpendicular to plane): $I = \frac{1}{2} MR^2$
- Annular disc (ring with inner radius r and outer radius R): $I = \frac{1}{2} M(R^2 + r^2)$
- Thin rod (axis perpendicular at midpoint): $I = \frac{Ml^2}{12}$
- Solid cylinder (about axis): $I = \frac{1}{2} MR^2$
- Hollow cylinder (about long axis): $I = MR^2$
- Solid sphere (about diameter): $I = \frac{2}{5} MR^2$
- Hollow sphere (thin shell, about diameter): $I = \frac{2}{3} MR^2$
- Uniform rectangular lamina (about axis through center, perpendicular to plane):

$$I = \frac{M(l^2 + b^2)}{12}$$
- Elliptical disc (about center, perpendicular to plane): $I = \frac{M}{4} (a^2 + b^2)$
- Uniform cone (axis from vertex to base center): $I = \frac{3}{10} MR^2$
- Triangular lamina:
 - About base: $I_1 = \frac{Mh^2}{6}$
 - About height: $I_2 = \frac{Mb^2}{6}$
 - About hypotenuse: $I_3 = \frac{Mb^2 h^2}{6(b^2 + h^2)}$

Angular Momentum and Torque Relations

$$L = I\omega$$

$$\tau = I\alpha = \frac{dL}{dt}$$

From conservation of angular momentum:

$$\frac{I_1}{I_2} = \frac{\tau_1}{\tau_2}$$

Radius of Gyration Formula

For n particles at distances r_1, r_2, \dots, r_n from the axis:

$$K = \sqrt{\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}}$$

Solved Examples

Example 1: Calculate the moment of inertia of a solid sphere of mass 2 kg and radius 0.5 m about its diameter.

Solution:

Given: $M = 2$ kg, $R = 0.5$ m

Moment of inertia of solid sphere:

$$I = \frac{2}{5}MR^2 = \frac{2}{5} \times 2 \times (0.5)^2 = \frac{4}{5} \times 0.25 = 0.2 \text{ kg} \cdot \text{m}^2$$

Therefore, $I = 0.2 \text{ kg} \cdot \text{m}^2$.

Practice Set

Conceptual Questions:

- **Level 1:** Define moment of inertia and state its physical significance.
- **Level 2:** Explain the principle of conservation of angular momentum.

Application-based Question:

- **Level 3:** A solid cylinder of mass 10 kg and radius 0.3 m rotates about its central axis with angular velocity 20 rad/s. Calculate its rotational kinetic energy.

Answer Key

Conceptual Questions:

- **Level 1:** Moment of inertia is the measure of an object's resistance to change in its rotational motion about an axis. It depends on mass distribution relative to the axis.
- **Level 2:** The principle states that if no external torque acts on a system, its total angular momentum remains constant.

Application-based Question:

Given:

$$M = 10 \text{ kg}, R = 0.3 \text{ m}, \omega = 20 \text{ rad/s}$$

Moment of inertia of solid cylinder:

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 10 \times (0.3)^2 = 5 \times 0.09 = 0.45 \text{ kg} \cdot \text{m}^2$$

Rotational kinetic energy:

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.45 \times (20)^2 = 0.225 \times 400 = 90 \text{ J}$$

Therefore, the rotational kinetic energy is 90 joules.

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