Binomial Theorem

The Binomial Theorem provides a method to expand expressions of the form $(a+b)^n$, where a and b are any real numbers and n is a positive integer. The expansion is a sum of terms involving powers of a and b multiplied by binomial coefficients.

Formula Derivation

The binomial expansion of $(a+b)^n$ is given by:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

where $\binom{n}{r}$ is the binomial coefficient defined as:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Each term in the expansion is called the $(r+1)^{th}$ term and is expressed as:

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Worked Illustrations

Example 1: Expand $(x+y)^3$.

Solution:

Using the binomial theorem,

$$(x+y)^3 = \sum_{r=0}^{3} \binom{3}{r} x^{3-r} y^r$$

Calculating each term:

- For $r = 0$: $\binom{3}{0} x^3 y^0 = 1 \times x^3 = x^3$
- For $r = 1$: $\binom{3}{1} x^2 y^1 = 3x^2 y$
- For $r = 2$: $\binom{3}{2} x^1 y^2 = 3xy^2$
- For $r = 3$: $\binom{3}{3} x^0 y^3 = y^3$

Therefore,

$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3$$

Solved Examples

Example 2: Find the middle term(s) in the expansion of $(2+x)^6$.

Solution:

Since $n=6$ is even, there is one middle term at position $\frac{n}{2}+1=4$.

The 4th term is:

$$T_4 = \binom{6}{3} 2^{6-3} x^3 = \binom{6}{3} 2^3 x^3$$

Calculate $\binom{6}{3}$:

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6\times 6} = 20$$

Calculate $2^3 = 8$.

Therefore,

$$T_4 = 20 \times 8 \times x^3 = 160x^3$$

The middle term is $160x^3$.

Practice Set

Level 1 – Easy

- Expand $(1+x)^4$.
- Find the coefficient of x^2 in $(3+x)^5$.
- Write the 3rd term in the expansion of $(x-2)^5$.

Level 2 – Moderate

- Find the middle term(s) in the expansion of $(1+2x)^7$.
- Find the term independent of x in the expansion of $(x+\frac{1}{x})^6$.
- Prove that the sum of the coefficients in the expansion of $(1+x)^n$ is 2^n.

Level 3 – Challenging

- Find the coefficient of x^5 in the expansion of $(2x-\frac{1}{x^2})^8$.
- Find the term containing x^3 in the expansion of $(1+3x)^9$.
- Using binomial theorem, prove that $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0$.

Answer Key

Level 1

- $(1+x)^4=1+4x+6x^2+4x^3+x^4$
- Coefficient of x^2 in $(3+x)^5$ is $\binom{5}{2}3^3=10\times 27=270$
- 3rd term in $(x-2)^5$ is $\binom{5}{2}x^3(-2)^2=10x^3\times 4=40x^3$

Level 2

- Middle term is the 4th term: $\binom{7}{3}(1)^4(2x)^3 = 35 \times 8x^3 = 280x^3$
- Term independent of x in $(x+\frac{1}{x})^6$ is the term where powers of x cancel out, i.e., $6-2r=0 \Rightarrow r=3$. Term: $\binom{6}{3}x^3(\frac{1}{x})^3 = 20$
- Sum of coefficients in $(1+x)^n$ is $\sum_{r=0}^{n}\binom{n}{r} = (1+1)^n = 2^n$

Level 3

- Coefficient of x^5 in $(2x-\frac{1}{x^2})^8$: Let term be $T_{r+1} = \binom{8}{r}(2x)^{8-r}(-\frac{1}{x^2})^r$. Power of x is $8-r-2r = 8-3r$. Set $8-3r=5 \Rightarrow r=1$. Coefficient: $\binom{8}{1}2^7(-1)^1 = 8 \times 128 \times (-1) = -1024$
- Term containing x^3 in $(1+3x)^9$: $T_{r+1} = \binom{9}{r}1^{9-r}(3x)^r$. Power of x is r. Set $r=3$. Term: $\binom{9}{3}3^3x^3 = 84 \times 27x^3 = 2268x^3$
- Proof: $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = (1-1)^n = 0$

Quick Reference

Term Number	General Term T_{r+1}	Binomial Coefficient $\binom{n}{r}$
$r+1$	$\binom{n}{r}a^{n-r}b^r$	$\frac{n!}{r!(n-r)!}$

Properties of Binomial Coefficients:

- Sum of coefficients: $\sum_{r=0}^{n}\binom{n}{r} = 2^n$
- Alternating sum: $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$
- Sum of even and odd coefficients: $\sum_{r\ \text{even}}\binom{n}{r} = \sum_{r\ \text{odd}}\binom{n}{r} = 2^{n-1}$

Glossary

- **Binomial Expression:** An algebraic expression with two terms connected by + or -.
- **Binomial Coefficient $\binom{n}{r}$:** The coefficient of the r^{th} term in the expansion of $(a+b)^n$.

- **General Term:** The $(r+1)^{th}$ term in the binomial expansion.
- **Middle Term:** The term(s) in the middle of the expansion, depending on whether n is even or odd.
- **Independent Term:** Term free from variables a and b in the expansion.