Sequence, Series and Arithmetic Mean

A **sequence** is a function whose domain is a subset of natural numbers, representing an ordered list of terms $f_1, f_2, f_3, \ldots, f_n$ where $f_n = f(n)$. A **real sequence** is one whose range is a subset of real numbers $\mathbb{R}$.

A **series** is the sum of the terms of a sequence, expressed as $a_1 + a_2 + a_3 + \ldots + a_n$. A **progression** is a sequence following a specific rule.

Series can be *finite* (with a finite number of terms) or *infinite* (with infinitely many terms).

Arithmetic Mean (A.M.)

When three quantities a, A, b are in arithmetic progression (A.P.), the middle term A is called the **arithmetic mean** between a and b. This means:

$$A - a = b - A$$

Adding A to both sides and rearranging, we get:

$$2A = a + b \implies A = \frac{a+b}{2}$$

Thus, the arithmetic mean between two numbers is half their sum.

n Arithmetic Means Between Two Numbers

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means inserted between two numbers a and b. The common difference d of the resulting A.P. is:

$$d = \frac{b-a}{n+1}$$

The arithmetic means are then:

$$A_k = a + kd = a + k \times \frac{b-a}{n+1}, \quad k = 1, 2, \ldots, n$$

Worked Example

Find 3 arithmetic means between 4 and 19.

Solution:

Number of means $n = 3$, $a = 4$, $b = 19$.

Calculate common difference:

$$d = \frac{19 - 4}{3 + 1} = \frac{15}{4} = 3.75$$

Arithmetic means:

- $A_1 = 4 + 1 \times 3.75 = 7.75$
- $A_2 = 4 + 2 \times 3.75 = 11.5$
- $A_3 = 4 + 3 \times 3.75 = 15.25$

Practice Set

- **Level 1 – Easy:** Find the arithmetic mean between 10 and 20.
- **Level 2 – Moderate:** Insert 4 arithmetic means between 7 and 27.
- **Level 3 – Challenging:** Find 5 arithmetic means between -3 and 12, and verify the sum of these means.

Answer Key

- Level 1: $\frac{10+20}{2} = 15$
- Level 2: $d = \frac{27-7}{5} = 4$, means: 11, 15, 19, 23
- Level 3: $d = \frac{12-(-3)}{6} = \frac{15}{6} = 2.5$, means: $-0.5, 2, 4.5, 7, 9.5$, sum = 22.5

Quick Reference

Concept	Formula
Arithmetic Mean between a and b	$\frac{a+b}{2}$
n Arithmetic Means between a and b	$A_k = a + k \times \frac{b-a}{n+1}$

Glossary

- **Sequence:** Ordered list of numbers.
- **Series:** Sum of sequence terms.
- **Arithmetic Progression (A.P.):** Sequence with constant difference.
- **Arithmetic Mean (A.M.):** Middle term in A.P. between two numbers.

Geometric Progression and Sum to n Terms of Special Series

A **geometric progression (G.P.)** is a sequence where each term after the first is obtained by multiplying the previous term by a constant called the *common ratio* r. Formally, for terms $a_1, a_2, \ldots, a_n$,

$$\frac{a_{k+1}}{a_k} = r, \quad \text{constant}$$

The sequence can be written as:

$$a, ar, ar^2, ar^3, \ldots$$

General Term of a G.P.

The n^{th} term T_n of a G.P. with first term a and common ratio r is:

$$T_n = ar^{n-1}$$

Term from the End

For a finite G.P. with n terms and last term l, the m^{th} term from the end is the $(n-m+1)^{th}$ term from the beginning:

$$T_{n-m+1} = l\left(\frac{1}{r}\right)^{m-1}$$

Sum of First n Terms of a G.P.

The sum S_n of the first n terms is given by:

For $r \neq 1$,

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & |r| < 1 \\ \frac{a(r^n-1)}{r-1}, & r > 1 \end{cases}$$

Infinite Sum of a G.P.

If $|r| < 1$, the infinite sum is:

$$S_\infty = \frac{a}{1-r}$$

Geometric Mean (G.M.)

The geometric mean between two positive numbers a and b is:

$$G = \sqrt{ab}$$

If a, b, c are in G.P., then:

$$b^2 = ac$$

Important Properties

- Reciprocals of terms in a G.P. form a G.P.
- If $G_1, G_2, \ldots, G_n$ are inserted between a and b in G.P., then:

$$G_k = a\left(\frac{b}{a}\right)^{\frac{k+1}{n+1}}, \quad 1 \leq k \leq n$$

- Product of terms equidistant from beginning and end equals $a \times l$.
- Raising each term of a G.P. to a power preserves the G.P. property.
- For three terms in G.P., choose $a/r, a, ar$.
- For four terms in G.P., choose $a/r^3, a/r, ar, ar^3$.
- For five terms in G.P., choose $a/r^2, a/r, a, ar, ar^2$.

Worked Example

Find the sum of the first 5 terms of a G.P. where $a = 3$ and $r = 2$.

Solution:

Using the formula for sum of first n terms:

$$S_5 = \frac{3(2^5 - 1)}{2 - 1} = \frac{3(32 - 1)}{1} = 3 \times 31 = 93$$

Practice Set

- **Level 1 – Easy:** Find the 4th term of the G.P. 2, 6, 18, ...
- **Level 2 – Moderate:** Find the sum of first 6 terms of a G.P. with $a = 5$ and $r = 3$.
- **Level 3 – Challenging:** Insert 3 geometric means between 2 and 162.

Answer Key

- Level 1: $T_4 = 2 \times 3^3 = 54$
- Level 2: $S_6 = \frac{5(3^6-1)}{3-1} = \frac{5(729-1)}{2} = \frac{5\times728}{2} = 1820$
- Level 3: Common ratio $r = \sqrt[4]{\frac{162}{2}} = 3$, means: 6, 18, 54

Quick Reference

Concept	Formula
n-th term of G.P.	$T_n = ar^{n-1}$
Sum of first n terms $(r \neq 1)$	$S_n = \frac{a(1-r^n)}{1-r}$ if $\|r\| < 1$, else $S_n = \frac{a(r^n-1)}{r-1}$
Sum to infinity $(\|r\| < 1)$	$S_\infty = \frac{a}{1-r}$
Geometric mean between a and b	$\sqrt{ab}$

Glossary

- **Geometric Progression (G.P.):** Sequence with constant ratio.
- **Common Ratio:** Constant multiplier between terms in G.P.
- **Geometric Mean (G.M.):** Square root of product of two numbers.
- **Sum to n terms:** Total of first n terms of a series.