Conic Sections Overview

Conic sections are curves obtained by intersecting a plane with a double-napped cone at different angles. The four primary types of conic sections are circle, parabola, ellipse, and hyperbola. Each conic section is defined as the locus of points satisfying specific distance properties relative to a fixed point called the focus and a fixed line called the directrix. The constant ratio of these distances is called the eccentricity, denoted by e.

Definitions:

- **Focus (S):** A fixed point used in the definition of conic sections.
- **Directrix:** A fixed straight line used in the definition of conic sections.
- **Eccentricity (e):** The constant ratio of the distance of any point on the conic from the focus to its distance from the directrix.

Depending on the value of eccentricity e, the conic sections are classified as follows:

- $e = 0$: Circle
- $e = 1$: Parabola
- $e < 1$: Ellipse
- $e > 1$: Hyperbola

Key terms related to conics:

- **Axis:** The straight line passing through the focus and perpendicular to the directrix.
- **Vertex:** The point of intersection of the conic and its axis.
- **Latus Rectum:** A line segment perpendicular to the axis passing through the focus, with endpoints on the conic.
- **Centre:** The point which bisects every chord of the conic passing through it.

Worked Illustration: Formation of Conic Sections

By varying the angle of the intersecting plane with respect to the axis of the double-napped cone, different conic sections are formed:

- Plane perpendicular to the axis: Circle
- Plane parallel to a generator of the cone: Parabola
- Plane intersecting both nappes but not parallel to the axis: Hyperbola
- Plane intersecting one nappe at an angle less than the angle of the cone: Ellipse

Practice Set

Level 1 – Easy

- Define the eccentricity of a conic section.
- Identify the conic section for $e = 0$.

Level 2 – Moderate

- Explain the role of the directrix in the definition of a parabola.
- Describe the difference between ellipse and hyperbola in terms of distances from foci.

Level 3 – Challenging

- Prove that the eccentricity of an ellipse is less than 1.
- Derive the equation of the axis of a conic section given its focus and directrix.

Answer Key

- **Eccentricity:** The ratio of the distance of any point on the conic from the focus to its distance from the directrix.
- **Conic for $e = 0$:** Circle.
- **Directrix in parabola:** A fixed line such that the distance from any point on the parabola to the directrix equals the distance to the focus.
- **Ellipse vs Hyperbola:** Ellipse: sum of distances to foci is constant; Hyperbola: difference of distances to foci is constant.
- **Proof of $e < 1$ for ellipse:** Since $c < a$, $e = \frac{c}{a} < 1$.
- **Equation of axis:** Line through focus perpendicular to directrix.

Quick Reference

Conic	Eccentricity (e)	Definition
Circle	0	Points equidistant from center
Parabola	1	Points equidistant from focus and directrix
Ellipse	<1	Sum of distances from two foci is constant
Hyperbola	>1	Difference of distances from two foci is constant

Glossary

- **Focus:** Fixed point used in conic definitions.
- **Directrix:** Fixed line used in conic definitions.
- **Eccentricity:** Ratio defining the shape of conics.
- **Axis:** Line through focus perpendicular to directrix.
- **Vertex:** Intersection of conic and axis.
- **Latus Rectum:** Line segment through focus perpendicular to axis.
- **Centre:** Midpoint of chord bisecting conic.

Circle

A circle is the locus of points in a plane equidistant from a fixed point called the center. The constant distance is called the radius.

Formula Derivation

Let the center be $C(h, k)$ and radius r. For any point $P(x, y)$ on the circle, the distance $CP = r$.

Using the distance formula:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Squaring both sides:

$$(x-h)^2 + (y-k)^2 = r^2$$

Worked Illustration

Find the equation of a circle with center $(3, -2)$ and radius 5.

Solution:

Using the formula:

$$(x-3)^2+(y+2)^2=25$$

Standard and General Equations

Standard form:

$$(x-h)^2+(y-k)^2=r^2$$

General form:

$$x^2+y^2+2gx+2fy+c=0$$

where $g=-h$, $f=-k$, and $c=h^2+k^2-r^2$.

Special Cases

- Circle passing through origin: $x^2+y^2-2hx-2ky=0$
- Center on x-axis: $(x-h)^2+y^2=r^2$
- Center on y-axis: $x^2+(y-k)^2=r^2$
- Circle touching axes: equations adjusted accordingly.

Practice Set

Level 1 – Easy

- Write the equation of a circle with center at origin and radius 4.
- Find the radius of the circle $x^2 + y^2 = 16$.

Level 2 – Moderate

- Find the center and radius of the circle $x^2 + y^2 - 6x + 8y + 9 = 0$.
- Write the equation of a circle touching the x-axis at (3,0) with radius 3.

Level 3 – Challenging

- Find the equation of the circle with diameter endpoints $(1, 2)$ and $(5, 6)$.
- Determine whether the point $(4, 1)$ lies inside, on, or outside the circle $(x - 2)^2 + (y + 3)^2 = 25$.

Answer Key

- Level 1: $x^2 + y^2 = 16$, radius = 4
- Level 2: Center $(3, -4)$, radius = 2; Equation: $(x - 3)^2 + y^2 = 9$
- Level 3: Equation: $(x - 3)^2 + (y - 4)^2 = 8$; Point lies inside if $(4 - 2)^2 + (1 + 3)^2 < 25$

Quick Reference

Property	Formula
Standard Equation	$(x - h)^2 + (y - k)^2 = r^2$
General Equation	$x^2 + y^2 + 2gx + 2fy + c = 0$
Radius	$r = \sqrt{h^2 + k^2 - c}$

Glossary

- **Center:** Fixed point equidistant from all points on the circle.
- **Radius:** Distance from center to any point on the circle.
- **Diameter:** Chord passing through the center.
- **Chord:** Line segment joining two points on the circle.

Parabola

A parabola is the locus of points equidistant from a fixed point called the focus and a fixed line called the directrix.

Formula Derivation

Let the focus be $F(h, k)$ and the directrix be the line $y = d$. For any point $P(x, y)$ on the parabola, the distance to the focus equals the distance to the directrix:

$$\sqrt{(x-h)^2 + (y-k)^2} = |y-d|$$

Squaring both sides and simplifying yields the standard equation of the parabola.

Standard Forms

- Parabola opening upwards/downwards: $(x-h)^2 = 4a(y-k)$
- Parabola opening right/left: $(y-k)^2 = 4a(x-h)$

Properties

- Vertex: Point (h, k)
- Focus: $(h, k+a)$ for vertical parabola

- Directrix: $y = k - a$ for vertical parabola
- Axis: Line through vertex and focus
- Latus Rectum length: $4a$

Worked Example

Find the equation of a parabola with vertex at origin and focus at $(0, 3)$.

Solution:

Distance $a = 3$. Since focus is above vertex, parabola opens upwards.

$$(x - 0)^2 = 4 \times 3 \times (y - 0) \implies x^2 = 12y$$

Practice Set

Level 1 – Easy

- Write the equation of a parabola with vertex at origin and focus at $(0, 2)$.
- Identify the vertex and focus of $y^2 = 8x$.

Level 2 – Moderate

- Find the equation of a parabola with vertex at $(1, 2)$ and focus at $(1, 5)$.
- Determine the length of the latus rectum for $(x - 2)^2 = 16(y + 3)$.

Level 3 – Challenging

- Prove that the parabola $y^2 = 4ax$ is the locus of points equidistant from focus and directrix.
- Find the coordinates of the focus and equation of directrix for $(y+1)^2 = -8(x-3)$.

Answer Key

- Level 1: $x^2 = 8y$; vertex (0,0), focus (2,0)
- Level 2: Equation: $(x-1)^2 = 6(y-2)$; latus rectum length = 16
- Level 3: Proof involves distance formula; focus at (3, -1), directrix $x = 7$

Quick Reference

Property	Equation
Parabola opening up/down	$(x-h)^2 = 4a(y-k)$
Parabola opening left/right	$(y-k)^2 = 4a(x-h)$
Focus	$(h, k+a)$ or $(h+a, k)$
Directrix	$y = k - a$ or $x = h - a$

Glossary

- **Focus:** Fixed point inside the parabola.
- **Directrix:** Fixed line used to define parabola.
- **Vertex:** Point where parabola changes direction.
- **Axis:** Line through vertex and focus.
- **Latus Rectum:** Chord through focus perpendicular to axis.

Ellipse

An ellipse is the locus of points in a plane such that the sum of the distances from two fixed points called foci is constant.

Formula Derivation

Let the foci be $S_1(-c, 0)$ and $S_2(c, 0)$, and the sum of distances to any point $P(x, y)$ on the ellipse be $2a$.

$$PS_1 + PS_2 = 2a$$

Using the distance formula:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

From this, the standard equation of the ellipse is derived as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2 - c^2$.

Terms Related to Ellipse

- **Foci:** Points S_1 and S_2 at distance $2c$.
- **Centre:** Midpoint of line segment joining foci.
- **Major axis:** Line segment through foci, length $2a$.
- **Minor axis:** Line segment perpendicular to major axis, length $2b$.
- **Vertices:** Endpoints of major axis.

- **Eccentricity:** $e = \frac{c}{a} < 1$.
- **Latus Rectum:** Line segment through focus perpendicular to major axis, length $\frac{2b^2}{a}$.

Standard Equations

- Horizontal ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$
- Vertical ellipse: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$

Worked Example

Find the equation of an ellipse with foci at $(\pm 3, 0)$ and major axis length 8.

Solution:

Given $c = 3$, $2a = 8 \Rightarrow a = 4$.

Calculate b:

$$b^2 = a^2 - c^2 = 16 - 9 = 7$$

Equation:

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

Practice Set

Level 1 – Easy

- Write the equation of an ellipse with center at origin, $a = 5, b = 3$.
- Find the eccentricity of ellipse with $a = 5, b = 4$.

Level 2 – Moderate

- Find the coordinates of foci for $\frac{x^2}{25} + \frac{y^2}{9} = 1$.
- Calculate the length of latus rectum for ellipse with $a = 6, b = 4$.

Level 3 – Challenging

- Prove that the sum of distances from any point on ellipse to foci is constant.
- Derive the equation of ellipse with vertical major axis.

Answer Key

- Level 1: Equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$; eccentricity $e = \sqrt{1 - \frac{16}{25}} = 0.6$
- Level 2: Foci at $(\pm 4, 0)$; latus rectum length $= \frac{2b^2}{a} = \frac{2 \times 16}{6} = \frac{32}{6}$
- Level 3: Proof uses definition of ellipse; derivation involves coordinate substitution.

Quick Reference

Property	Horizontal Ellipse	Vertical Ellipse
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Eccentricity	$e = \frac{c}{a}$	$e = \frac{c}{a}$

Latus Rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$

Glossary

- **Foci:** Two fixed points defining ellipse.
- **Major axis:** Longest diameter through foci.
- **Minor axis:** Diameter perpendicular to major axis.
- **Eccentricity:** Ratio c/a measuring ellipse shape.
- **Latus Rectum:** Chord through focus perpendicular to major axis.

Hyperbola

A hyperbola is the locus of points in a plane such that the difference of the distances from two fixed points called foci is constant.

Formula Derivation

Let the foci be $S_1(-c, 0)$ and $S_2(c, 0)$, and the difference of distances to any point $P(x, y)$ on the hyperbola be $2a$.

$$|PS_2 - PS_1| = 2a$$

Using the distance formula:

$$|\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2}| = 2a$$

From this, the standard equation of the hyperbola is derived as:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = c^2 - a^2$.

Terms Related to Hyperbola

- **Foci:** Points S_1 and S_2 at distance $2c$.
- **Centre:** Midpoint of line segment joining foci.
- **Transverse axis:** Line through foci, length $2a$.
- **Conjugate axis:** Line through centre perpendicular to transverse axis, length $2b$.
- **Vertices:** Points where hyperbola intersects transverse axis.
- **Eccentricity:** $e = \frac{c}{a} > 1$.
- **Directrix:** Lines $x = \pm\frac{a^2}{c}$.
- **Latus Rectum:** Line segment through focus perpendicular to transverse axis, length $\frac{2b^2}{a}$.

Standard Equations

- Hyperbola with transverse axis along x-axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Conjugate hyperbola with transverse axis along y-axis: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Worked Example

Find the equation of a hyperbola with foci at $(\pm 5, 0)$ and vertices at $(\pm 3, 0)$.

Solution:

Given $c = 5, a = 3$.

Calculate b:

$$b^2 = c^2 - a^2 = 25 - 9 = 16$$

Equation:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Practice Set

Level 1 – Easy

- Write the equation of a hyperbola with center at origin, $a = 4, b = 3$.
- Find the eccentricity of hyperbola with $a = 3, b = 4$.

Level 2 – Moderate

- Find the coordinates of foci for $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
- Calculate the length of latus rectum for hyperbola with $a = 5, b = 12$.

Level 3 – Challenging

- Prove that the difference of distances from any point on hyperbola to foci is constant.

- Derive the equation of conjugate hyperbola.

Answer Key

- Level 1: Equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$; eccentricity $e = \sqrt{1 + \frac{9}{16}} = 1.25$
- Level 2: Foci at $(\pm 5, 0)$; latus rectum length $= \frac{2b^2}{a} = \frac{2\times 144}{5} = 57.6$
- Level 3: Proof uses definition of hyperbola; conjugate hyperbola equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Quick Reference

Property	Hyperbola	Conjugate Hyperbola
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Eccentricity	$e = \frac{c}{a}$	$e = \frac{c}{a}$
Latus Rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$

Glossary

- **Foci:** Two fixed points defining hyperbola.
- **Transverse axis:** Axis through vertices and foci.
- **Conjugate axis:** Axis perpendicular to transverse axis.
- **Eccentricity:** Ratio c/a greater than 1.
- **Directrix:** Lines $x = \pm\frac{a^2}{c}$.
- **Latus Rectum:** Chord through focus perpendicular to transverse axis.