Limit and Its Fundamentals

Concept Explanation:

The limit of a function describes the value that the function approaches as the input approaches a particular point. For a function $f(x)$, the limit as x approaches a is denoted by

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ gets arbitrarily close to L as x approaches a (but $x \neq a$).

The left-hand limit (LHL) and right-hand limit (RHL) are defined as:

- Left-hand limit: $\lim_{x \to a^-} f(x) = L_1$, the value approached as x approaches a from the left.
- Right-hand limit: $\lim_{x \to a^+} f(x) = L_2$, the value approached as x approaches a from the right.

The limit $\lim_{x \to a} f(x)$ exists if and only if $L_1 = L_2 = L$.

Formula Derivation

For polynomial functions $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, the limit is

$$\lim_{x \to a} f(x) = a_0 + a_1a + a_2a^2 + \cdots + a_na^n = f(a)$$

For rational functions $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials and $h(a) \neq 0$,

$$\lim_{x \to a} f(x) = \frac{g(a)}{h(a)}$$

If $g(a) = 0$ and $h(a) = 0$, factorization and cancellation are used to evaluate the limit.

Worked Illustrations

Example: Find $\lim_{x \to 1} (x-1)^2$.

Left-hand limit:

Set $x = 1 - h$, $h \to 0^+$, then

$$\lim_{h \to 0^+} (1 - h - 1)^2 = \lim_{h \to 0^+} (-h)^2 = 0$$

Right-hand limit:

Set $x = 1 + h$, $h \to 0^+$, then

$$\lim_{h \to 0^+} (1 + h - 1)^2 = \lim_{h \to 0^+} h^2 = 0$$

Since LHL = RHL = 0, the limit exists and equals 0.

Solved Examples

Example 1: Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution:

Using the standard limit,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Example 2: Find $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution:

Factor numerator:

$$x^2 - 4 = (x-2)(x+2)$$

Cancel $x - 2$:

$$\lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 4$$

Practice Set

- **Level 1 – Easy**
- Find $\lim_{x \to 3} (2x+1)$.
- Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$.
- **Level 2 – Moderate**
- Calculate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.
- Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.
- **Level 3 – Challenging**

- Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.
- Find $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$.

Answer Key

- Level 1
- $\lim_{x \to 3}(2x + 1) = 2(3) + 1 = 7$
- $\lim_{x \to 0} \frac{\sin 3x}{x} = 3$ (using $\lim_{x \to 0} \frac{\sin x}{x} = 1$)
- Level 2
- $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1}(x^2 + x + 1) = 3$
- $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
- Level 3
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- Rationalize numerator:

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

Quick Reference

Concept	Formula
Limit of polynomial	$\lim_{x \to a} f(x) = f(a)$
Limit of rational function	$\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$, if $h(a) \neq 0$
Standard limit	$\lim_{x \to 0} \frac{\sin x}{x} = 1$
Left-hand limit	$\lim_{x \to a^-} f(x)$
Right-hand limit	$\lim_{x \to a^+} f(x)$

Glossary

- **Limit:** The value a function approaches as the input approaches a point.
- **Left-hand limit:** Limit as input approaches from the left side.
- **Right-hand limit:** Limit as input approaches from the right side.
- **Indeterminate form:** Expressions like $\frac{0}{0}$ that require algebraic manipulation to evaluate limits.
- **Rational function:** A function expressed as the ratio of two polynomials.

Derivatives

Concept Explanation:

The derivative of a function $f(x)$ at a point a measures the instantaneous rate of change of the function at that point. It is defined as the limit of the difference quotient:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Formula Derivation

Starting from the definition, the derivative at a is the slope of the tangent line to the curve $y = f(x)$ at $x = a$.

For a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$, the derivative is

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

Worked Illustrations

Example: Find the derivative of $f(x) = \frac{1}{x}$ using the definition.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Solved Examples

Example 1: Find $\frac{d}{dx}(x^3 + 2x)$.

Solution:

Using power rule:

$$\frac{d}{dx}x^3 = 3x^2, \quad \frac{d}{dx}2x = 2$$

Therefore,

$$\frac{d}{dx}(x^3 + 2x) = 3x^2 + 2$$

Example 2: Differentiate $f(x) = \sin x$.

Solution:

$$\frac{d}{dx}(\sin x) = \cos x$$

Practice Set

- **Level 1 – Easy**
- Find $\frac{d}{dx}(5x^2)$.
- Differentiate $f(x) = 3x + 7$.
- **Level 2 – Moderate**
- Find $\frac{d}{dx}(x^3 - 4x)$.
- Differentiate $f(x) = \cos x$.
- **Level 3 – Challenging**
- Find $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ using the definition.
- Differentiate $f(x) = x^n$ where n is any real number.

Answer Key

- Level 1
- $\frac{d}{dx}(5x^2) = 10x$
- $\frac{d}{dx}(3x + 7) = 3$
- Level 2
- $\frac{d}{dx}(x^3 - 4x) = 3x^2 - 4$
- $\frac{d}{dx}(\cos x) = -\sin x$
- Level 3
- Using definition,

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-2x - h}{x^2(x+h)^2} = -\frac{2}{x^3}$$

- $\frac{d}{dx}(x^n) = nx^{n-1}$

Quick Reference

Derivative Rule	Formula
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$
Sum Rule	$(u + v)' = u' + v'$
Product Rule	$(uv)' = u'v + uv'$
Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$
Derivative of $\sin x$	$\cos x$
Derivative of $\cos x$	$-\sin x$

Prepzy

Glossary

- **Derivative:** Instantaneous rate of change of a function at a point.
- **Difference quotient:** $\frac{f(a+h) - f(a)}{h}$, average rate of change over interval h.
- **Power rule:** Derivative of x^n is nx^{n-1}.
- **Product rule:** Derivative of product uv is $u'v + uv'$.
- **Quotient rule:** Derivative of quotient $\frac{u}{v}$ is $\frac{vu' - uv'}{v^2}$.