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Definition of Polynomials

A polynomial is an algebraic expression consisting of variables and coefficients, involving only the operations of addition, subtraction, multiplication, and non-negative integer exponents of variables. It can be expressed in the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number called the degree of the polynomial.

Types of polynomials based on number of terms:

- **Monomial:** A polynomial with one term, e.g., $5x^3$.
- **Binomial:** A polynomial with two terms, e.g., $x^2 + 3x$.
- **Trinomial:** A polynomial with three terms, e.g., $x^2 + 5x + 6$.

Worked Example

Identify whether the following are polynomials and classify them:

- $4x^3 - 2x + 7$
- $3x^{-2} + 5$
- $2x^2 + 3x + 1$

Solution:

- $4x^3 - 2x + 7$ is a polynomial of degree 3 with three terms (trinomial).

- $3x^{-2} + 5$ is not a polynomial because of the negative exponent.
- $2x^2 + 3x + 1$ is a polynomial of degree 2 with three terms (trinomial).

Practice Set

- *Level 1:* Identify if $7x^4 + 3x^2 - 5$ is a polynomial.
- *Level 2:* Classify $5x^3 - 4x + 1$ as monomial, binomial, or trinomial.
- *Level 3:* Determine if $2x^{1/2} + 3$ is a polynomial and justify.

Answer Key

- Level 1: Yes, it is a polynomial of degree 4.
- Level 2: Trinomial (three terms).
- Level 3: Not a polynomial because exponent is not a whole number.

Quick Reference

- Polynomial: sum of terms with variables raised to whole number exponents.
- Monomial: one term.
- Binomial: two terms.
- Trinomial: three terms.

Glossary

- **Polynomial:** Algebraic expression with non-negative integer exponents.
- **Degree:** Highest exponent of variable in polynomial.
- **Monomial:** Single term polynomial.
- **Binomial:** Polynomial with two terms.
- **Trinomial:** Polynomial with three terms.

Standard Form of Polynomials

The standard form of a polynomial arranges its terms in descending order of degree. For example, a polynomial $p(x)$ of degree n is written as:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$.

Worked Example

Write the polynomial $3 + 4x^3 - 2x + x^2$ in standard form.

Solution:

Arrange terms from highest to lowest degree:

$$4x^3 + x^2 - 2x + 3$$

Practice Set

- *Level 1:* Write $5x + 2x^2 + 7$ in standard form.
- *Level 2:* Arrange $6 - 3x^4 + x^3$ in standard form.
- *Level 3:* Express $x - 4 + 2x^5 - 3x^2$ in standard form.

Answer Key

- Level 1: $2x^2 + 5x + 7$
- Level 2: $-3x^4 + x^3 + 6$

- Level 3: $2x^5 - 3x^2 + x - 4$

Quick Reference

- Standard form: terms ordered by descending powers of x.

Glossary

- **Standard form:** Polynomial with terms arranged from highest to lowest degree.

Degree of a Polynomial

The degree of a polynomial is the highest power of the variable in the polynomial expression.

For example, in $p(x) = 4x^5 + 3x^3 - x + 7$, the degree is 5.

Significance

- Determines the shape and behavior of the polynomial graph.
- Indicates the maximum number of zeros the polynomial can have.

Worked Example

Find the degree of $2x^4 - 5x^2 + 3x - 1$.

Solution: The highest power of x is 4, so degree is 4.

Practice Set

- *Level 1:* Degree of $7x^3 + 2x + 1$.
- *Level 2:* Degree of $5x^7 - 3x^5 + x^2$.
- *Level 3:* Degree of $4x^0 + 6x^1 - 2x^3 + x^4$.

Answer Key

- Level 1: 3
- Level 2: 7
- Level 3: 4

Quick Reference

- Degree = highest exponent of variable.

Glossary

- **Degree:** Highest power of variable in polynomial.

Types of Polynomials Based on Degree

Polynomials are classified by their degree as follows:

- **Linear Polynomial:** Degree 1, e.g., $ax + b$.
- **Quadratic Polynomial:** Degree 2, e.g., $ax^2 + bx + c$.
- **Cubic Polynomial:** Degree 3, e.g., $ax^3 + bx^2 + cx + d$.

Worked Example

Classify $3x^3 + 2x^2 - x + 5$.

Solution: Degree is 3, so it is a cubic polynomial.

Practice Set

- *Level 1:* Identify the type of $4x + 7$.
- *Level 2:* Identify the type of $5x^2 - 3x + 1$.
- *Level 3:* Identify the type of $2x^4 + x^3 - 5$.

Answer Key

- Level 1: Linear polynomial.
- Level 2: Quadratic polynomial.
- Level 3: Polynomial of degree 4 (quartic).

Quick Reference

- Linear: degree 1
- Quadratic: degree 2
- Cubic: degree 3
- Quartic: degree 4

Glossary

- **Linear Polynomial:** Degree 1 polynomial.
- **Quadratic Polynomial:** Degree 2 polynomial.
- **Cubic Polynomial:** Degree 3 polynomial.

Polynomial Operations

Operations on polynomials include addition, subtraction, multiplication, and division.

Addition and Subtraction

Combine like terms (terms with the same variable and exponent).

Example: Add $(3x^2 + 2x + 1) + (5x^2 - x + 4)$

Step 1: Group like terms:

$$(3x^2 + 5x^2) + (2x - x) + (1 + 4)$$

Step 2: Add coefficients:

$$8x^2 + x + 5$$

Multiplication

Multiply each term of the first polynomial by each term of the second polynomial and combine like terms.

Example: Multiply $(x + 2)(x + 3)$

Step 1: Multiply terms:

$$x \times x = x^2$$

$x \times 3 = 3x$

$2 \times x = 2x$

$2 \times 3 = 6$

Step 2: Add like terms:

$x^2 + 3x + 2x + 6 = x^2 + 5x + 6$

Division

Polynomial division is performed similar to long division of numbers.

Practice Set

- *Level 1:* Add $(2x^2 + 3x) + (x^2 - x + 4)$.
- *Level 2:* Subtract $(5x^3 + 2x) - (3x^3 + x)$.
- *Level 3:* Multiply $(x + 4)(x^2 - x + 1)$.

Answer Key

- Level 1: $3x^2 + 2x + 4$
- Level 2: $2x^3 + x$
- Level 3: $x^3 - x^2 + x + 4x^2 - 4x + 4 = x^3 + 3x^2 - 3x + 4$

Quick Reference

- Add/subtract by combining like terms.

- Multiply each term by every term in the other polynomial.
- Divide using polynomial long division.

Glossary

- **Like terms:** Terms with the same variable and exponent.
- **Polynomial multiplication:** Distributive multiplication of terms.
- **Polynomial division:** Dividing polynomials similar to numbers.

Factorisation of Polynomials

Factorisation is expressing a polynomial as a product of its factors.

Common Methods

- **Factor by grouping:** Group terms and factor common factors.
- **Use of identities:** Apply formulas like $a^2 - b^2 = (a - b)(a + b)$.
- **Quadratic factorisation:** For quadratic $ax^2 + bx + c$, find factors of ac that sum to b.

Worked Example

Factorise $x^2 + 5x + 6$.

Step 1: Find two numbers that multiply to 6 and add to 5: 2 and 3.

Step 2: Write as $x^2 + 2x + 3x + 6$.

Step 3: Group terms: $(x^2 + 2x) + (3x + 6)$.

Step 4: Factor each group: $x(x + 2) + 3(x + 2)$.

Step 5: Factor common binomial: $(x + 2)(x + 3)$.

Practice Set

- *Level 1:* Factorise $x^2 - 9$.
- *Level 2:* Factorise $x^2 + 7x + 12$.
- *Level 3:* Factorise $2x^2 + 7x + 3$.

Answer Key

- Level 1: $(x - 3)(x + 3)$
- Level 2: $(x + 3)(x + 4)$
- Level 3: $(2x + 1)(x + 3)$

Quick Reference

- Factor by grouping or use identities.
- For quadratics, find factors of ac summing to b.

Glossary

- **Factorisation:** Expressing polynomial as product of factors.
- **Factor theorem:** If $f(a) = 0$, then $x - a$ is a factor.

Factor Theorem

The Factor Theorem states that for a polynomial $f(x)$, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Proof

By polynomial division, dividing $f(x)$ by $(x - a)$ gives:

$$f(x) = (x - a)q(x) + r$$

where $q(x)$ is the quotient and r is the remainder (a constant).

Substitute $x = a$:

$$f(a) = (a - a)q(a) + r = 0 + r = r$$

If $f(a) = 0$, then $r = 0$, so $(x - a)$ divides $f(x)$ exactly.

Worked Example

Check if $(x - 2)$ is a factor of $f(x) = x^3 - 4x^2 + 5x - 2$.

Calculate $f(2)$:

$2^3 - 4(2)^2 + 5(2) - 2 = 8 - 16 + 10 - 2 = 0$

Since $f(2) = 0$, $(x - 2)$ is a factor.

Practice Set

- *Level 1:* Verify if $(x-1)$ is a factor of x^2-3x+2.
- *Level 2:* Check if $(x+3)$ is a factor of $x^3+2x^2-9x-18$.
- *Level 3:* Determine if $(x-4)$ is a factor of $2x^3-3x^2-8x+12$.

Answer Key

- Level 1: $f(1)=1-3+2=0$, so factor.
- Level 2: $f(-3)=-27+18+27-18=0$, so factor.
- Level 3: $f(4)=128-48-32+12=60\neq 0$, not a factor.

Quick Reference

- If $f(a)=0$, then $(x-a)$ is a factor.

Glossary

- **Factor Theorem:** Connects zeros of polynomial to factors.

Zeros of Polynomials

A zero (root) of a polynomial $p(x)$ is a value k such that $p(k)=0$.

Geometrically, zeros correspond to x-intercepts of the graph of $y=p(x)$.

Properties:

- A polynomial of degree n has at most n zeros.
- Zeros can be real or complex.

Worked Example

Find zeros of $p(x) = x^2 - 5x + 6$.

Factorise:

$x^2 - 5x + 6 = (x - 2)(x - 3)$

Zeros are $x = 2$ and $x = 3$.

Practice Set

- *Level 1:* Find zeros of $x^2 - 4$.
- *Level 2:* Find zeros of $x^2 + 3x + 2$.
- *Level 3:* Find zeros of $2x^3 - 3x^2 - 2x + 3$.

Answer Key

- Level 1: $x = 2, -2$
- Level 2: $x = -1, -2$
- Level 3: $x = 1, \frac{3}{2}, -1$

Quick Reference

- Zeros satisfy $p(x) = 0$.
- Number of zeros ≤ degree of polynomial.

Glossary

- **Zero of polynomial:** Value where polynomial equals zero.

Graphing Polynomials

The graph of a polynomial $p(x)$ is a curve in the coordinate plane.

Key features:

- Degree determines the general shape.
- Leading coefficient affects end behavior.
- Zeros correspond to x-intercepts.
- For quadratic polynomials $ax^2 + bx + c$, the graph is a parabola.

Quadratic Graph Details

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Axis of symmetry: $x = -\frac{b}{2a}$

Direction: Opens upward if $a > 0$, downward if $a < 0$.

Discriminant $D = b^2 - 4ac$

- $D > 0$: Two distinct real roots, parabola crosses x-axis twice.
- $D = 0$: One real root, parabola touches x-axis.
- $D < 0$: No real roots, parabola does not intersect x-axis.

Worked Example

Graph $y = x^2 - 4x + 3$.

Calculate vertex:

$x = -\frac{-4}{2 \times 1} = 2$

$y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$

Vertex at $(2, -1)$, parabola opens upward.

Discriminant:

$D = (-4)^2 - 4(1)(3) = 16 - 12 = 4 > 0$

Two distinct real roots.

Practice Set

- *Level 1:* Find vertex and axis of symmetry of $y = x^2 + 6x + 8$.
- *Level 2:* Determine number of roots of $y = 2x^2 - 4x + 2$.
- *Level 3:* Sketch graph of $y = -x^2 + 4x - 3$.

Answer Key

- Level 1: Vertex at $(-3, -1)$, axis $x = -3$.
- Level 2: $D = (-4)^2 - 4(2)(2) = 16 - 16 = 0$, one real root.
- Level 3: Parabola opens downward, vertex at $(2, 1)$.

Quick Reference

- Vertex formula: $x = -\frac{b}{2a}$.
- Discriminant determines roots.
- Leading coefficient sign determines parabola direction.

Glossary

- **Vertex:** Highest or lowest point of parabola.
- **Axis of symmetry:** Vertical line through vertex.
- **Discriminant:** Determines nature of roots.

Applications of Polynomials

Polynomials model various real-world phenomena in physics, engineering, economics, and other fields.

- **Physics:** Motion equations, projectile paths.
- **Engineering:** Signal processing, control systems.
- **Economics:** Cost and revenue functions.
- **Traffic flow:** Modeling and optimization.

Worked Example

In economics, profit $P(x)$ can be modeled as a polynomial $P(x) = -2x^2 + 40x - 100$, where x is the number of units sold.

Find the number of units x that maximizes profit.

Step 1: Find vertex $x = -\frac{b}{2a} = -\frac{40}{2\times(-2)} = 10$.

Step 2: Maximum profit at $x = 10$ units.

Practice Set

- *Level 1:* Identify degree of polynomial modeling cost $C(x) = 5x + 100$.
- *Level 2:* Find zeros of $R(x) = x^2 - 9$ representing revenue.
- *Level 3:* For $P(x) = -3x^2 + 12x - 9$, find maximum profit and units sold.

Answer Key

- Level 1: Degree 1 (linear).
- Level 2: Zeros at $x = 3, -3$.
- Level 3: Vertex at $x = 2$, maximum profit $P(2) = -3(4) + 24 - 9 = 3$.

Quick Reference

- Polynomials model real-world relationships.
- Vertex gives maxima or minima in quadratic models.

Glossary

- **Application:** Use of polynomials to model real situations.
- **Maximum/Minimum:** Highest or lowest value of polynomial function.