Pair of Linear Equations in Two Variables

A pair of linear equations in two variables consists of two equations each of the form $ax + by + c = 0$, where a, b, c are real constants and a and b are not both zero. The general form is:

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

Concept Explanation

Each equation represents a straight line on the Cartesian plane. The solution to the pair is the set of points (x, y) that satisfy both equations simultaneously.

Formula Derivation

To solve the pair, we seek values of x and y satisfying both equations. The nature of solutions depends on the ratios:

$$\frac{a_1}{a_2}, \quad \frac{b_1}{b_2}, \quad \frac{c_1}{c_2}$$

- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the system has a unique solution (lines intersect).
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, infinitely many solutions (coincident lines).
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, no solution (parallel lines).

Worked Illustration

Consider the pair:

$$3x - y + 7 = 0 \quad \text{and} \quad 7x + y = 3$$

Rearranged as:

$$3x - y = -7 \quad \text{and} \quad 7x + y = 3$$

Adding both equations:

$$(3x + 7x) + (-y + y) = -7 + 3 \Rightarrow 10x = -4 \Rightarrow x = -\frac{2}{5}$$

Substitute x into first equation:

$$3\left(-\frac{2}{5}\right) - y = -7 \Rightarrow -\frac{6}{5} - y = -7 \Rightarrow -y = -7 + \frac{6}{5} = -\frac{29}{5} \Rightarrow y = \frac{29}{5}$$

Solution: $x = -\frac{2}{5}, y = \frac{29}{5}$

Solved Example

Example: Solve the pair by substitution method:

$$7x - 15y = 2 \quad (i)$$

$$x + 2y = 3 \quad (ii)$$

Solution:

From (ii), express x in terms of y:

$$x = 3 - 2y$$

Substitute into (i):

$$7(3 - 2y) - 15y = 2 \Rightarrow 21 - 14y - 15y = 2 \Rightarrow -29y = -19 \Rightarrow y = \frac{19}{29}$$

Substitute y back into $x = 3 - 2y$:

$$x = 3 - 2 \times \frac{19}{29} = 3 - \frac{38}{29} = \frac{49}{29}$$

Solution: $x = \frac{49}{29}, y = \frac{19}{29}$

Practice Set

Level 1 – Easy

- Solve $x + y = 5$ and $x - y = 1$ by substitution method.
- Find the solution of $2x + 3y = 12$ and $x - y = 1$ by elimination method.

Level 2 – Moderate

- Solve $3x - 2y = 7$ and $4x + y = 1$ using substitution method.
- Find the solution of $5x + 2y = 14$ and $3x - 4y = 2$ using elimination method.

Level 3 – Challenging

- Determine the nature of solutions for $2x + 3y = 6$ and $4x + 6y = 12$.
- Solve $6x - 9y = 15$ and $2x - 3y = 5$ and interpret the result.

Answer Key

- Level 1:
  - $x = 3, y = 2$
  - $x = 3, y = 2$
- Level 2:
  - $x = 1, y = -2$
  - $x = 2, y = 2$
- Level 3:
  - Infinite solutions (lines coincide).
  - Infinite solutions (second equation is multiple of first).

Quick Reference

Condition	Ratio of Coefficients	Nature of Solutions
Unique Solution	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Lines intersect at one point
Infinite Solutions	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Lines coincide
No Solution	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Lines are parallel

Glossary

- **Variable:** Symbol representing an unknown quantity.
- **Coefficient:** Numerical factor multiplying a variable.
- **Linear Equation:** Equation of the first degree in variables.
- **Consistent System:** System with at least one solution.
- **Inconsistent System:** System with no solution.
- **Dependent System:** System with infinitely many solutions.

Graphical Method for Solving Pair of Linear Equations

The graphical method involves plotting the lines represented by the two linear equations on the Cartesian plane and identifying their point(s) of intersection.

Concept Explanation

Each linear equation in two variables represents a straight line. The solution to the pair corresponds to the point(s) where the lines intersect.

Formula Derivation

Rewrite each equation in the form $y = mx + c$ where m is the slope and c is the y-intercept.

Plot at least two points for each line by choosing values of x and calculating corresponding y values.

Draw the lines and observe their intersection.

Worked Illustration

Given equations:

$$y = 2x - 2 \quad \text{and} \quad y = 4x - 4$$

For $y = 2x - 2$, values:

x	y
0	-2
1	0
2	2

For $y = 4x - 4$, values:

x	y
0	-4
1	0

Plotting these points and drawing lines, the lines intersect at $(1, 0)$.

Solved Example

Example: Find the solution of the pair:

$$y = 2x - 2 \quad \text{and} \quad y = 4x - 4$$

Solution:

Plot points as above and draw lines. The intersection point is $(1, 0)$, so $x = 1, y = 0$ is the solution.

Practice Set

Level 1 – Easy

- Graph and find the solution of $y = x + 1$ and $y = -x + 3$.
- Plot $y = 2x$ and $y = 2x + 1$ and determine the nature of solutions.

Level 2 – Moderate

- Graph $3x + 2y = 6$ and $6x + 4y = 12$ and find the solution.
- Plot $y = -x + 2$ and $y = -x + 5$ and interpret the result.

Level 3 – Challenging

- Graph $2x - 3y = 6$ and $4x - 6y = 10$ and analyze the solution.

- Plot $y = \frac{1}{2}x + 1$ and $y = \frac{1}{2}x + 1$ and explain the solution set.

Answer Key

- Level 1:
  - Solution at $(1, 2)$
  - No solution (parallel lines)
- Level 2:
  - Infinite solutions (coincident lines)
  - No solution (parallel lines)
- Level 3:
  - No solution (parallel lines)
  - Infinite solutions (coincident lines)

Quick Reference

Graph Type	Condition	Solution
Intersecting Lines	Lines cross at one point	Unique solution
Coincident Lines	Lines overlap	Infinite solutions
Parallel Lines	Lines never meet	No solution

Glossary

- **Slope:** Rate of change of y with respect to x.
- **Y-intercept:** Point where line crosses the y-axis.
- **Graphical Solution:** Finding solutions by plotting lines.

Algebraic Methods for Solving Pair of Linear Equations

Algebraic methods include substitution and elimination techniques to find the solution of a pair of linear equations.

Concept Explanation

These methods transform the system into a single-variable equation to solve for one variable, then back-substitute to find the other.

Formula Derivation

Substitution Method

1. Express one variable in terms of the other from one equation.
2. Substitute this expression into the second equation.
3. Solve the resulting single-variable equation.
4. Back-substitute to find the other variable.

Elimination Method

1. Multiply equations to equalize coefficients of one variable.
2. Add or subtract equations to eliminate that variable.
3. Solve the resulting single-variable equation.
4. Back-substitute to find the other variable.

Solved Examples

Example 1 (Substitution):

Given:

$$7x - 15y = 2 \quad (i)$$

$$x + 2y = 3 \quad (ii)$$

From (ii): $x = 3 - 2y$

Substitute into (i):

$$7(3 - 2y) - 15y = 2 \Rightarrow 21 - 14y - 15y = 2 \Rightarrow -29y = -19 \Rightarrow y = \frac{19}{29}$$

Back-substitute:

$$x = 3 - 2 \times \frac{19}{29} = \frac{49}{29}$$

Solution: $x = \frac{49}{29}, y = \frac{19}{29}$

Example 2 (Elimination):

Given:

$$2x + 3y = 8 \quad (i)$$

$$4x + 6y = 7 \quad (ii)$$

Multiply (i) by 2:

$$4x + 6y = 16 \quad (iii)$$

Subtract (ii) from (iii):

$$(4x + 6y) - (4x + 6y) = 16 - 7 \Rightarrow 0 = 9$$

This is a contradiction, so no solution exists.

Practice Set

Level 1 – Easy

- Solve $x + y = 4$ and $x - y = 2$ by substitution.
- Solve $3x + 2y = 12$ and $6x + 4y = 24$ by elimination.

Level 2 – Moderate

- Solve $5x - y = 9$ and $3x + 2y = 7$ by substitution.
- Solve $4x + 5y = 20$ and $2x + 3y = 11$ by elimination.

Level 3 – Challenging

- Determine the solution nature of $2x + 3y = 6$ and $4x + 6y = 10$.
- Solve $7x - 2y = 3$ and $14x - 4y = 6$ and interpret the result.

Answer Key

- Level 1:
  - $x = 3, y = 1$
  - Infinite solutions (dependent equations)
- Level 2:
  - $x = 2, y = 1$
  - $x = 1, y = 4$
- Level 3:
  - No solution (inconsistent system)
  - Infinite solutions (dependent system)

Quick Reference

Method	Steps
Substitution	Express one variable, substitute, solve, back-substitute
Elimination	Equalize coefficients, add/subtract, solve, back-substitute

Glossary

- **Substitution Method:** Solving by replacing one variable with an expression.
- **Elimination Method:** Solving by adding or subtracting equations to eliminate a variable.
- **Consistent System:** Has at least one solution.
- **Inconsistent System:** Has no solution.
- **Dependent System:** Has infinitely many solutions.