Trigonometric Ratios and their values

In a right-angled triangle, the trigonometric ratios relate the angles to the lengths of the sides. Consider a right triangle ABC, right-angled at B, with angle $\theta = \angle BAC$. The sides are named as follows:

- Hypotenuse (H) = side opposite the right angle (AC)
- Base (B) = side adjacent to angle θ (AB)
- Perpendicular (P) = side opposite to angle θ (BC)

The six trigonometric ratios of angle θ are defined as:

- $\sin\theta = \frac{P}{H} = \frac{BC}{AC}$
- $\cos\theta = \frac{B}{H} = \frac{AB}{AC}$
- $\tan\theta = \frac{P}{B} = \frac{BC}{AB}$
- $\cot\theta = \frac{B}{P} = \frac{AB}{BC}$
- $\sec\theta = \frac{H}{B} = \frac{AC}{AB}$
- $\csc\theta = \frac{H}{P} = \frac{AC}{BC}$

Note that $\csc\theta$, $\sec\theta$, and $\cot\theta$ are reciprocals of $\sin\theta$, $\cos\theta$, and $\tan\theta$ respectively. Also, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

The values of these ratios depend only on the angle θ, not on the size of the triangle.

Worked Illustration

Consider the following values of trigonometric ratios for common angles:

Angle (θ)	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	\sqrt{3}	\text{Not defined}
$\cot\theta$	\text{Not defined}	\sqrt{3}	1	\frac{1}{\sqrt{3}}	0
$\sec\theta$	1	\frac{2}{\sqrt{3}}	\sqrt{2}	2	\text{Not defined}
$\csc\theta$	\text{Not defined}	2	\sqrt{2}	\frac{2}{\sqrt{3}}	1

Solved Example 1

Given $\tan A = \frac{4}{3}$, find the other trigonometric ratios of angle A.

Solution:

Let the perpendicular $P = 4a$ and base $B = 3a$. Using Pythagoras theorem, hypotenuse H is:

$$H = \sqrt{P^2 + B^2} = \sqrt{(4a)^2 + (3a)^2} = \sqrt{16a^2 + 9a^2} = \sqrt{25a^2} = 5a.$$

Now, calculate the ratios:

- $\sin A = \frac{P}{H} = \frac{4a}{5a} = \frac{4}{5}$
- $\cos A = \frac{B}{H} = \frac{3a}{5a} = \frac{3}{5}$
- $\cot A = \frac{1}{\tan A} = \frac{3}{4}$
- $\sec A = \frac{1}{\cos A} = \frac{5}{3}$
- $\csc A = \frac{1}{\sin A} = \frac{5}{4}$

Practice Set

Level 1 – Easy

- Find $\sin 30^\circ$, $\cos 60^\circ$, and $\tan 45^\circ$.
- In a right triangle, if $\sin\theta = \frac{3}{5}$, find $\cos\theta$ and $\tan\theta$.

Level 2 – Moderate

- Given $\tan\theta = 2$, find all other trigonometric ratios.
- In a right triangle, the base is 7 cm and the hypotenuse is 25 cm. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

Level 3 – Challenging

- Prove that $\sin\theta + \cos\theta = 1$ is not possible for any acute angle θ.
- Find the value of θ if $\tan\theta = \frac{5}{12}$ and calculate all other ratios.

Answer Key

Level 1

- $\sin 30^\circ = \frac{1}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 45^\circ = 1$
- $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

Level 2

- $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\cot\theta = \frac{1}{2}$, $\sec\theta = \sqrt{5}$, $\csc\theta = \frac{\sqrt{5}}{2}$
- Perpendicular $P = \sqrt{25^2 - 7^2} = 24$ cm, $\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$

Level 3

- $\sin\theta + \cos\theta = 1$ is false for acute angles because $\sin^2\theta + \cos^2\theta = 1$ and both $\sin\theta$ and $\cos\theta$ are positive and less than 1.
- Using Pythagoras theorem, hypotenuse $H = 13a$, $P = 5a$, $B = 12a$. Ratios: $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\cot\theta = \frac{12}{5}$, $\sec\theta = \frac{13}{12}$, $\csc\theta = \frac{13}{5}$

Quick Reference

- $\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$
- $\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$
- $\tan\theta = \frac{\text{Perpendicular}}{\text{Base}}$
- $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$
- $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}}$
- $\csc\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$

Glossary

- **Hypotenuse:** The longest side of a right-angled triangle, opposite the right angle.
- **Base:** The side adjacent to the angle of interest in a right triangle.
- **Perpendicular:** The side opposite the angle of interest in a right triangle.
- **Acute Angle:** An angle less than 90 degrees.
- **Trigonometric Ratios:** Ratios of sides of a right triangle relative to an angle.

Trigonometric Identities

A trigonometric identity is an equation involving trigonometric ratios that holds true for all values of the variable within its domain.

Consider a right triangle ABC, right-angled at B. By Pythagoras theorem:

$$AB^2 + BC^2 = AC^2$$

Dividing both sides by AC^2,

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

Or,

$$\cos^2 A + \sin^2 A = 1$$

This is the fundamental Pythagorean identity.

Dividing the original equation by AB^2,

$$1 + \tan^2 A = \sec^2 A$$

Dividing the original equation by BC^2,

$$\cot^2 A + 1 = \csc^2 A$$

These identities hold for angles $0^\circ \leq A \leq 90^\circ$, with domain restrictions where functions are undefined.

Worked Illustration

Express $\cos A$, $\tan A$, and $\sec A$ in terms of $\sin A$.

Solution:

From the identity,

$$\cos^2 A = 1 - \sin^2 A$$

$$\Rightarrow \cos A = \pm\sqrt{1 - \sin^2 A}$$

Taking the positive root for acute angles,

$$\cos A = \sqrt{1 - \sin^2 A}$$

Then,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}$$

Solved Example 2

Prove that

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{1}{\sec\theta - \tan\theta}$$

Solution:

Start with LHS:

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1}$$

Rewrite numerator and denominator in terms of $\tan\theta$ and $\sec\theta$:

$$\sin\theta = \tan\theta\cos\theta, \quad \cos\theta = \cos\theta$$

Multiply numerator and denominator by $\frac{1}{\cos\theta}$:

$$\frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 + \sec\theta} = \frac{\tan\theta + \sec\theta - 1}{\tan\theta + \sec\theta + 1}$$

Multiply numerator and denominator by $\tan\theta - \sec\theta$:

$$\frac{(\tan\theta + \sec\theta - 1)(\tan\theta - \sec\theta)}{(\tan\theta + \sec\theta + 1)(\tan\theta - \sec\theta)}$$

Expand numerator:

$$(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)$$

Using identity $\sec^2\theta - \tan^2\theta = 1$, numerator becomes:

$$-1 - \tan\theta + \sec\theta$$

Denominator simplifies to:

$$(\tan\theta + \sec\theta)^2 - 1^2 = (\tan\theta + \sec\theta)^2 - 1$$

After simplification, the expression equals:

$$\frac{-1}{\tan\theta - \sec\theta} = \frac{1}{\sec\theta - \tan\theta} = \text{RHS}$$

Practice Set

Level 1 – Easy

- Verify the identity $\sin^2\theta + \cos^2\theta = 1$ for $\theta = 30^\circ$.
- Express $\sec\theta$ in terms of $\tan\theta$ using the identity $1 + \tan^2\theta = \sec^2\theta$.

Level 2 – Moderate

- Prove that $1 + \cot^2\theta = \csc^2\theta$.
- Express $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Level 3 – Challenging

- Prove the identity $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{1}{\sec\theta - \tan\theta}$.
- Derive the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ using Pythagorean identities.

Answer Key

Level 1

- $\sin^2 30^\circ + \cos^2 30^\circ = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$
- $\sec\theta = \sqrt{1 + \tan^2\theta}$

Level 2

- Using Pythagorean theorem, $\cot^2\theta + 1 = \csc^2\theta$ is true by dividing sides by $\sin^2\theta$.
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Level 3

- Proof provided in solved example 2.
- Using $\cos^2\theta + \sin^2\theta = 1$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ follows by definition of double angle.

Quick Reference

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Glossary

- **Identity:** An equation true for all values of the variable within its domain.
- **Pythagoras Theorem:** In a right triangle, the square of the hypotenuse equals the sum of squares of the other two sides.
- **Secant ($\sec\theta$):** Reciprocal of cosine.
- **Cosecant ($\csc\theta$):** Reciprocal of sine.
- **Cotangent ($\cot\theta$):** Reciprocal of tangent.