- Polynomials
- Remainder Theorem
- Factor Theorem
- Algebraic Identities

Polynomials

A polynomial is an algebraic expression in which the variables involved have only non-negative integral exponents. A polynomial in one variable x is expressed in the standard form as:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real coefficients with $a_n \neq 0$, and n is the degree of the polynomial.

Types of Polynomials

- **Constant Polynomial:** Degree 0 polynomial, e.g., 4, $-\frac{7}{5}$, $\frac{3}{4}$.
- **Zero Polynomial:** The zero polynomial has all coefficients zero and its degree is undefined.
- **Monomial:** Polynomial with one non-zero term, e.g., $2x$, $-4x^2$.
- **Binomial:** Polynomial with two non-zero terms, e.g., $4x^2 + 8$.
- **Trinomial:** Polynomial with three non-zero terms, e.g., $x^2 + 2x + 4$.
- **Linear Polynomial:** Degree 1 polynomial, $ax + b$, $a \neq 0$.
- **Quadratic Polynomial:** Degree 2 polynomial, $ax^2 + bx + c$, $a \neq 0$.
- **Cubic Polynomial:** Degree 3 polynomial, $ax^3 + bx^2 + cx + d$, $a \neq 0$.

Degree of a Polynomial

The degree is the highest power of the variable in the polynomial. For multivariable polynomials, the degree is the highest sum of exponents in any term.

Zeroes of a Polynomial

A zero of polynomial $p(x)$ is a number c such that $p(c) = 0$. The maximum number of zeroes equals the degree of the polynomial.

Worked Example

Verify whether 3 and 0 are zeroes of $p(x) = x^2 - 3x$.

Solution:

Calculate $p(3) = 3^2 - 3 \times 3 = 9 - 9 = 0$, so 3 is a zero.

Calculate $p(0) = 0^2 - 3 \times 0 = 0$, so 0 is a zero.

Practice Set

- *Level 1:* Identify the degree and type of polynomials: $5x^3 - 2x + 7$, 4, $x^2 + 3x + 1$.
- *Level 2:* Find zeroes of $p(x) = x^2 - 5x + 6$.
- *Level 3:* Determine the degree and zeroes of $7x^3 - 4x^2y + 3xy^2 - y^3$.

Answer Key

- Level 1: Degrees 3, 0, 2 respectively; types cubic, constant, quadratic.
- Level 2: Zeroes are 2 and 3.
- Level 3: Degree 3; zeroes depend on variables.

Quick Reference

Polynomial Type	Degree	Example
Constant	0	4
Linear	1	3x + 6
Quadratic	2	x^2 + 4x + 4

Cubic	3	2x^3 - 4x^2 + 8x - 3

Glossary

- **Polynomial:** Algebraic expression with non-negative integral exponents.
- **Degree:** Highest power of variable(s) in polynomial.
- **Zero of Polynomial:** Value making polynomial zero.
- **Monomial, Binomial, Trinomial:** Polynomials with 1, 2, 3 terms respectively.

Remainder Theorem

The Remainder Theorem states that when a polynomial $p(x)$ is divided by a linear divisor $x - a$, the remainder is $p(a)$.

Formally, if $p(x)$ and $g(x)$ are polynomials with $\deg p(x) \geq \deg g(x)$ and $g(x) \neq 0$, then:

$$p(x) = g(x) \times q(x) + r(x)$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Worked Example

Divide $p(x) = 3x^2 + x - 1$ by $g(x) = x + 1$ using polynomial long division.

Solution:

Divide $3x^2$ by x to get $3x$. Multiply $3x(x + 1) = 3x^2 + 3x$. Subtract to get remainder $-2x - 1$.

Divide $-2x$ by x to get -2. Multiply $-2(x + 1) = -2x - 2$. Subtract to get remainder 1.

So, $3x^2 + x - 1 = (x + 1)(3x - 2) + 1$.

Worked Example

Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$ using the Remainder Theorem.

Solution:

Zero of divisor $x - 1 = 0 \Rightarrow x = 1$.

Calculate $p(1) = 1 + 1 - 2 + 1 + 1 = 2$.

Remainder is 2.

Practice Set

- *Level 1:* Find remainder when $x^3 + 2x^2 + 3x + 4$ is divided by $x - 2$.
- *Level 2:* Divide $2x^3 - 3x^2 + 4x - 5$ by $x + 1$ and find quotient and remainder.
- *Level 3:* Use polynomial long division to divide $x^4 - 1$ by $x^2 - 1$.

Answer Key

- Level 1: Remainder $p(2) = 8 + 8 + 6 + 4 = 26$.
- Level 2: Quotient $2x^2 - 5x + 9$, remainder -14.
- Level 3: Quotient $x^2 + 1$, remainder 0.

Quick Reference

Step	Action
1	Divide leading term of dividend by leading term of divisor.
2	Multiply divisor by quotient term.
3	Subtract product from dividend.
4	Bring down next term and repeat.

Glossary

- **Dividend:** Polynomial to be divided.
- **Divisor:** Polynomial by which division is done.
- **Quotient:** Result of division.
- **Remainder:** Leftover polynomial after division.

Factor Theorem

The Factor Theorem states that $(x - a)$ is a factor of polynomial $p(x)$ if and only if $p(a) = 0$.

Key points:

- If remainder $r(x) = 0$ when $p(x)$ is divided by $x - a$, then $x - a$ is a factor.
- For polynomial $ax^2 + bx + c$, factors are $(x - \alpha)(x - \beta)$ where $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Factorisation by Splitting the Middle Term

For quadratic $x^2 + lx + m$, find a, b such that $a + b = l$ and $ab = m$. Then:

$$x^2 + lx + m = x^2 + ax + bx + ab = x(x + a) + b(x + a) = (x + a)(x + b)$$

Worked Example

Factorise $6x^2 + 17x + 5$ by splitting the middle term and using Factor Theorem.

Solution:

Splitting method: Find a, b such that $a + b = 17$ and $ab = 30$. Choose $2, 15$.

$6x^2 + 17x + 5 = 6x^2 + 2x + 15x + 5 = 2x(3x + 1) + 5(3x + 1) = (2x + 5)(3x + 1)$.

Factor Theorem: Check zeroes of $p(x) = x^2 + \frac{17}{6}x + \frac{5}{6}$. $p(-\frac{1}{3}) = 0$, so $x + \frac{1}{3}$ is a factor.

Similarly, $x + \frac{5}{2}$ is a factor. Hence, $6x^2 + 17x + 5 = (3x + 1)(2x + 5)$.

Practice Set

- *Level 1:* Factorise $x^2 + 5x + 6$ by splitting the middle term.
- *Level 2:* Use Factor Theorem to factorise $x^3 + 6x^2 + 11x + 6$.
- *Level 3:* Factorise $2x^3 - 3x^2 - 2x + 3$ using Factor Theorem.

Answer Key

- Level 1: $(x + 2)(x + 3)$.
- Level 2: $(x + 1)(x + 2)(x + 3)$.
- Level 3: $(x - 1)(2x + 3)(x - 1)$ or similar factorisation.

Quick Reference

Step	Action
1	Find factors of constant term.
2	Test factors as zeroes using $p(x)$.
3	Divide polynomial by factor $(x - a)$.
4	Factor quotient further if possible.

Glossary

- **Factor Theorem:** $x - a$ is factor if $p(a) = 0$.
- **Splitting Middle Term:** Express middle term as sum of two terms to factorise.

Algebraic Identities

An algebraic identity is an equation true for all values of variables involved.

Common identities include:

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $x^2 - y^2 = (x+y)(x-y)$
- $(x+a)(x+b) = x^2 + (a+b)x + ab$
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$
- $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$
- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$
- $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
- $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Worked Example

Factorise $x^2 + 4xy + 4y^2$.

Solution:

Using identity $(x+y)^2 = x^2 + 2xy + y^2$, here $4xy = 2 \times x \times 2y$, so:

$$x^2 + 4xy + 4y^2 = (x+2y)^2 = (x+2y)(x+2y)$$

Worked Example

Evaluate $(102)^3$ using algebraic identities.

Solution:

Express $102 = 100 + 2$. Using $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$:

$$(102)^3 = (100+2)^3 = 100^3 + 2^3 + 3 \times 100 \times 2 \times (100+2)$$

Calculate:

$$= 1,000,000 + 8 + 3 \times 100 \times 2 \times 102 = 1,000,000 + 8 + 61,200 = 1,061,208$$

Practice Set

- *Level 1:* Expand $(x+3)^2$ and $(x-5)^2$.
- *Level 2:* Factorise $x^3 - 27$ and $8x^3 + 1$.
- *Level 3:* Prove $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.

Answer Key

- Level 1: $x^2 + 6x + 9$, $x^2 - 10x + 25$.
- Level 2: $(x-3)(x^2+3x+9)$, $(2x+1)(4x^2-2x+1)$.
- Level 3: Proof involves expanding RHS and simplifying to LHS.

Quick Reference

Identity	Formula
Square of sum	$(x+y)^2 = x^2 + 2xy + y^2$
Difference of squares	$x^2 - y^2 = (x+y)(x-y)$
Sum of cubes	$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
Difference of cubes	$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Glossary

- **Algebraic Identity:** Equation true for all variable values.
- **Sum of Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.
- **Difference of Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.