Mathematically Acceptable Statements

A *statement* in mathematics is a sentence that is either true or false, but not both. It is neither a question, an order, nor an exclamation. Statements can be classified as:

- Always true
- Always false
- Ambiguous (not clearly true or false)

Examples:

- "The sun sets in the west." — Always true
- "There are 8 days in a week." — Always false
- "It is raining here." — Ambiguous (depends on location)
- "Dogs are intelligent." — Ambiguous (subjective)

In mathematics, statements must be unambiguous and either true or false. For example, "5 + 2 = 7" is true, while "5 + 3 = 7" is false.

Worked Examples

Determine whether the following statements are true, false, or ambiguous:

- There are 8 days in a week. **False** (7 days in a week)
- It is raining here. **Ambiguous** (location not specified)
- The sun sets in the west. **True**
- Gauri is a kind girl. **Ambiguous** (subjective)
- The product of two odd integers is even. **False** (product is odd)
- The product of two even natural numbers is even. **True** (proved later)

Practice Set

- **Level 1 (Easy):** Identify if the statement "The sum of two even numbers is even" is true or false.
- **Level 2 (Moderate):** Determine if "Every prime number greater than 2 is odd" is true or false.
- **Level 3 (Challenging):** Analyze the statement "The product of two consecutive integers is always even" and justify your answer.

Answer Key

- Level 1: True, sum of two even numbers is always even.
- Level 2: True, except 2, all primes are odd.
- Level 3: True, because among two consecutive integers, one must be even.

Quick Reference

- Mathematical statements are either true or false.
- Ambiguous statements are not accepted in mathematics.
- Counter-examples disprove false statements.

Glossary

- **Statement:** A sentence that is either true or false.
- **Ambiguous:** Not clearly true or false.
- **Counter-example:** An example that disproves a statement.

Deductive Reasoning

Deductive reasoning is the process of deriving a conclusion from known facts or statements using logical steps. It is used to establish the truth of mathematical statements.

Example puzzle: Given cards with letters and numbers, the rule is "If a card has an even number on one side, then it has a vowel on the other side." To verify this, you only need to turn over cards that could violate the rule.

- Turn over the card with an even number to check if the other side is a vowel.
- Turn over the card with a consonant to check if the other side is an even number (to disprove the rule).

Deductive reasoning helps conclude the truth of statements based on previously established truths.

Worked Example

Given the statement: "If a number is odd, then its square is odd." Using deductive reasoning, since the square of an odd number is always odd, the statement is true.

Practice Set

- **Level 1 (Easy):** Use deductive reasoning to prove "The sum of two even numbers is even."
- **Level 2 (Moderate):** Prove that "The product of two odd numbers is odd."

- **Level 3 (Challenging):** Using deductive reasoning, prove that "The sum of the interior angles of a triangle is 180°."

Answer Key

- Level 1: Sum of two even numbers is even because each even number is divisible by 2, so their sum is divisible by 2.
- Level 2: Product of two odd numbers is odd because odd numbers can be expressed as $2k + 1$, and their product is $(2k + 1)(2m + 1) = 4km + 2k + 2m + 1$, which is odd.
- Level 3: See detailed proof in the next section.

Quick Reference

- Deductive reasoning derives conclusions from known facts.
- It is essential for mathematical proofs.

Glossary

- **Deductive Reasoning:** Logical process of deriving conclusions from premises.
- **Transversal:** A line that intersects two or more lines.
- **Alternate Angles:** Angles on opposite sides of a transversal.

Theorems, Conjectures and Axioms

Theorem: A mathematical statement that has been proven to be true using logical reasoning.

Conjecture: A statement believed to be true based on observations but not yet proven.

Axiom: A self-evident truth accepted without proof, forming the foundation of mathematical reasoning.

Worked Examples

Theorem A1.1: The sum of the interior angles of a triangle is 180°.

Theorem A1.2: The product of two even natural numbers is even.

Conjecture Example: The sum of three consecutive even numbers is divisible by 6.

Practice Set

- **Level 1 (Easy):** Identify whether the statement "The sum of two odd numbers is even" is a theorem, conjecture, or axiom.
- **Level 2 (Moderate):** Formulate a conjecture based on the pattern of triangular numbers.
- **Level 3 (Challenging):** Explain why axioms are necessary in mathematics and give an example.

Answer Key

- Level 1: The statement is a theorem (can be proven).
- Level 2: The sum of the first n natural numbers is $\frac{n(n+1)}{2}$.
- Level 3: Axioms are necessary as starting points for proofs; for example, Euclid's postulate that a straight line can be drawn between any two points.

Quick Reference

- Theorems are proven statements.
- Conjectures are unproven but believed true.

- Axioms are accepted truths without proof.

Glossary

- **Theorem:** Proven mathematical statement.
- **Conjecture:** Unproven statement believed true.
- **Axiom:** Fundamental accepted truth.

Mathematical Proof

A *mathematical proof* is a logical argument that establishes the truth of a theorem beyond any doubt. Unlike verification by examples, a proof applies to all cases.

Difference between Verification and Proof:

- Verification checks a statement for some examples but cannot guarantee it is always true.
- Proof uses logical reasoning to show a statement is true in all cases.

Ingredients of a Proof

- Clear understanding of the hypothesis.
- Logical sequence of statements.
- Use of previously established theorems, axioms, or definitions.
- Conclusion matching the statement to be proved.

Worked Example: Proof of Theorem A1.1

Theorem: The sum of the interior angles of a triangle is 180°.

Proof:

1. Consider triangle ABC.
2. Draw line DE through A parallel to BC.
3. Since DE is parallel to BC and AB is a transversal, alternate angles are equal: $\angle DAB = \angle ABC$.
4. Similarly, $\angle CAE = \angle ACB$.
5. Sum of angles on line DE at A is 180°: $\angle DAB + \angle BAC + \angle CAE = 180^\circ$.
6. Substitute equal angles: $\angle ABC + \angle BAC + \angle ACB = 180^\circ$.

Practice Set

- **Level 1 (Easy):** Verify the sum of interior angles of a triangle using a drawn triangle.
- **Level 2 (Moderate):** Prove that the product of two even natural numbers is even.
- **Level 3 (Challenging):** Prove that the product of any three consecutive even natural numbers is divisible by 16.

Answer Key

- Level 1: Sum of interior angles measured approximately 180°.
- Level 2: Let $x = 2m$, $y = 2n$, then $xy = 4mn$ which is divisible by 2, so even.
- Level 3: Let the numbers be $2n$, $2n + 2$, $2n + 4$. Their product is $8n(n + 1)(n + 2)$. Since among three consecutive integers one is even, the product is divisible by 16.

Quick Reference

- Proofs establish universal truth.
- Use logical deductions and known facts.
- Counter-examples disprove statements.

Glossary

- **Proof:** Logical argument establishing truth.
- **Hypothesis:** Given information in a theorem.
- **Conclusion:** Statement to be proved.
- **Alternate Angles:** Equal angles formed by a transversal crossing parallel lines.