Introduction to Rational Numbers

In Mathematics, equations such as $x + 2 = 13$ are solved by finding the value of x that satisfies the equation. For example, $x = 11$ satisfies this equation. However, some equations require expanding the number system to find solutions. For instance, $x + 5 = 5$ has solution $x = 0$, which is a whole number but not a natural number. Similarly, $x + 18 = 5$ requires $x = -13$, an integer. Equations like $2x = 3$ and $5x + 7 = 0$ require rational numbers $\frac{3}{2}$ and $-\frac{7}{5}$ respectively.

This progression shows the need to understand rational numbers, which are numbers expressible as $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Properties of Rational Numbers

Closure Property

Closure property states that performing an operation on two numbers of a set results in a number of the same set.

- **Whole Numbers:** Closed under addition and multiplication but not under subtraction and division.
- **Integers:** Closed under addition, subtraction, and multiplication but not under division.
- **Rational Numbers:** Closed under addition, subtraction, and multiplication. Not closed under division by zero.

Commutativity

Commutativity means changing the order of numbers does not change the result.

- **Whole Numbers:** Addition and multiplication are commutative; subtraction and division are not.
- **Integers:** Addition and multiplication are commutative; subtraction and division are not.
- **Rational Numbers:** Addition and multiplication are commutative; subtraction and division are not.

Associativity

Associativity means changing the grouping of numbers does not change the result.

- **Whole Numbers:** Addition and multiplication are associative; subtraction and division are not.
- **Integers:** Addition and multiplication are associative; subtraction and division are not.
- **Rational Numbers:** Addition and multiplication are associative; subtraction and division are not.

Role of Zero (0)

Zero is the additive identity for whole numbers, integers, and rational numbers. For any number a,

$$a + 0 = 0 + a = a$$

Role of One (1)

One is the multiplicative identity for whole numbers, integers, and rational numbers. For any number a,

$$a \times 1 = 1 \times a = a$$

Distributivity of Multiplication over Addition and Subtraction

For all rational numbers a, b, and c,

$$a(b + c) = ab + ac \quad , \quad a(b - c) = ab - ac$$

Worked Illustrations and Solved Examples

Example 1: Addition of Rational Numbers

Find $\frac{3}{7} + \left(-\frac{6}{11}\right) + \left(-\frac{8}{21}\right) + \frac{5}{22}$.

Solution:

Find the LCM of denominators 7, 11, 21, and 22, which is 462.

Convert each fraction:

$$\frac{3}{7} = \frac{198}{462}, \quad -\frac{6}{11} = -\frac{252}{462}, \quad -\frac{8}{21} = -\frac{176}{462}, \quad \frac{5}{22} = \frac{105}{462}$$

Sum:

$$\frac{198 - 252 - 176 + 105}{462} = \frac{-125}{462}$$

Using commutativity and associativity, grouping can simplify calculations.

Example 2: Multiplication of Rational Numbers

Find $-\frac{4}{5} \times \frac{3}{7} \times \frac{15}{16} \times -\frac{14}{9}$.

Solution:

Group and multiply numerators and denominators:

$$\left(-\frac{4}{5} \times \frac{15}{16}\right) \times \left(\frac{3}{7} \times -\frac{14}{9}\right) = -\frac{12}{35} \times -\frac{35}{24} = \frac{420}{840} = \frac{1}{2}$$

Example 3: Using Distributivity

Find $\frac{2}{5} \times -\frac{3}{7} - \frac{1}{14} - \frac{3}{7} \times \frac{3}{5}$.

Solution:

Rewrite using commutativity:

$$\frac{2}{5} \times -\frac{3}{7} - \frac{3}{7} \times \frac{3}{5} - \frac{1}{14} = -\frac{3}{7} \times \left(\frac{2}{5} + \frac{3}{5}\right) - \frac{1}{14} = -\frac{3}{7} \times 1 - \frac{1}{14} = -\frac{3}{7} - \frac{1}{14} = -\frac{6}{14} - \frac{1}{14} = -\frac{7}{14} = -\frac{1}{2}$$

Practice Set

Level 1 – Easy

- Find $\frac{1}{3} + \frac{2}{3}$.
- Calculate $5 + 0$ and $0 + 5$.
- Verify if $4 \times 1 = 4$.

Level 2 – Moderate

- Simplify $\frac{3}{4} + \left(-\frac{1}{2}\right)$.
- Check if subtraction is commutative for $\frac{5}{6}$ and $\frac{1}{3}$.
- Calculate $\left(\frac{2}{5} + \frac{3}{10}\right) \times 2$ using distributivity.

Level 3 – Challenging

- Prove that multiplication is associative for rational numbers $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$.
- Solve $3x + 5 = 2x + 9$ where x is a rational number.
- Show that division is not associative using $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

Answer Key

Level 1

- $\frac{1}{3} + \frac{2}{3} = 1$
- $5 + 0 = 5$ and $0 + 5 = 5$
- $4 \times 1 = 4$

Level 2

- $\frac{3}{4} + \left(-\frac{1}{2}\right) = \frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$
- Subtraction is not commutative: $\frac{5}{6} - \frac{1}{3} = \frac{5}{6} - \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$, but $\frac{1}{3} - \frac{5}{6} = \frac{2}{6} - \frac{5}{6} = -\frac{3}{6} = -\frac{1}{2}$
- Using distributivity: $\left(\frac{2}{5} + \frac{3}{10}\right) \times 2 = \frac{2}{5} \times 2 + \frac{3}{10} \times 2 = \frac{4}{5} + \frac{6}{10} = \frac{4}{5} + \frac{3}{5} = \frac{7}{5}$

Level 3

- Associativity of multiplication:
$\left(\frac{1}{2} \times \frac{3}{4}\right) \times \frac{5}{6} = \frac{3}{8} \times \frac{5}{6} = \frac{15}{48} = \frac{5}{16}$
$\frac{1}{2} \times \left(\frac{3}{4} \times \frac{5}{6}\right) = \frac{1}{2} \times \frac{15}{24} = \frac{15}{48} = \frac{5}{16}$
- Solving $3x + 5 = 2x + 9$:
$3x - 2x = 9 - 5 \Rightarrow x = 4$
- Division is not associative:
$\left(\frac{1}{2} \div \frac{1}{3}\right) \div \frac{1}{4} = \frac{3}{2} \div \frac{1}{4} = \frac{3}{2} \times 4 = 6$
$\frac{1}{2} \div \left(\frac{1}{3} \div \frac{1}{4}\right) = \frac{1}{2} \div \left(\frac{1}{3} \times 4\right) = \frac{1}{2} \div \frac{4}{3} = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

Quick Reference

Property	Whole Numbers	Integers	Rational Numbers
Closure	Addition, Multiplication	Addition, Subtraction, Multiplication	Addition, Subtraction, Multiplication
Commutativity	Addition, Multiplication	Addition, Multiplication	Addition, Multiplication
Associativity	Addition, Multiplication	Addition, Multiplication	Addition, Multiplication
Identity Element	0 (Additive), 1 (Multiplicative)	0 (Additive), 1 (Multiplicative)	0 (Additive), 1 (Multiplicative)
Distributivity	Yes	Yes	Yes

Glossary

- **Rational Number:** A number expressible as $\frac{p}{q}$ where p and q are integers and $q \neq 0$.
- **Closure Property:** An operation is closed on a set if performing it on members of the set results in a member of the same set.
- **Commutative Property:** Changing the order of numbers does not change the result.
- **Associative Property:** Changing the grouping of numbers does not change the result.
- **Additive Identity:** The number 0, which when added to any number, leaves it unchanged.
- **Multiplicative Identity:** The number 1, which when multiplied by any number, leaves it unchanged.
- **Distributive Property:** Multiplication distributes over addition and subtraction.