Powers with Negative Exponents

Negative exponents represent the reciprocal of the base raised to the corresponding positive exponent. For any non-zero number a and positive integer m, the power with negative exponent is defined as:

$$a^{-m} = \frac{1}{a^m}$$

This means that a^{-m} is the multiplicative inverse of a^m.

Formula Derivation

Consider the pattern for powers of 10:

$$10^2 = 10 \times 10 = 100$$

$$10^1 = 10 = \frac{100}{10}$$

$$10^0 = 1 = \frac{10}{10}$$

$$10^{-1} = \frac{1}{10}$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

Extending this pattern, for any positive integer m,

$$10^{-m} = \frac{1}{10^m}$$

Similarly, for any non-zero base a,

$$a^{-m} = \frac{1}{a^m}$$

Worked Illustrations

- Calculate 2^{-2}:

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

- Calculate 3^{-3}:

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

Solved Examples

1. Find the multiplicative inverse of 2^{-4}.

Solution:

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

Therefore, the multiplicative inverse is 16.

Practice Set

Level 1 – Easy

1. Find the value of 10^{-3}.
2. Express 5^{-1} as a fraction.

Level 2 – Moderate

3. Calculate 4^{-2} and express as a decimal.
4. Find the multiplicative inverse of 7^{-3}.

Level 3 – Challenging

5. Simplify $(2^{-3} \times 3^{-2})$.
6. Prove that $a^{-m} = \frac{1}{a^m}$ for any non-zero integer a and positive integer m.

Answer Key

1. $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
2. $5^{-1} = \frac{1}{5} = 0.2$
3. $4^{-2} = \frac{1}{4^2} = \frac{1}{16} = 0.0625$
4. Multiplicative inverse of 7^{-3} is $7^3 = 343$
5. $2^{-3} \times 3^{-2} = \frac{1}{2^3} \times \frac{1}{3^2} = \frac{1}{8} \times \frac{1}{9} = \frac{1}{72}$
6. Proof involves the definition of negative exponents as reciprocals, as shown in the formula derivation section.

Quick Reference

Expression	Equivalent
a^{-m}	$\frac{1}{a^m}$
10^{-m}	$\frac{1}{10^m}$

Glossary

- **Negative Exponent:** An exponent that indicates the reciprocal of the base raised to the positive exponent.
- **Multiplicative Inverse:** A number which when multiplied by the original number gives 1.
- **Base:** The number that is multiplied by itself.
- **Exponent:** The number that indicates how many times the base is multiplied.

Laws of Exponents with Negative Integers

The laws of exponents extend to negative integers as exponents. For any non-zero integer a and integers m and n, the following laws hold:

1. Product of Powers

$$a^m \times a^n = a^{m+n}$$

This holds for all integers m and n, including negative integers.

2. Quotient of Powers

$$\frac{a^m}{a^n} = a^{m-n}$$

3. Power of a Power

$$(a^m)^n = a^{m \times n}$$

4. Power of a Product

$$(a^m)(b^m) = (ab)^m$$

5. Zero Exponent

$$a^0 = 1$$

for any non-zero a.

Formula Derivation

Using the definition of negative exponents as reciprocals, the laws are consistent with the addition and subtraction of exponents.

Solved Examples

1. Simplify $2^{-3} \times 2^{-2}$.

   Solution:

$$2^{-3} \times 2^{-2} = 2^{-3+(-2)} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

2. Simplify $5^{-2} \times 5^4$.

   Solution:

$$5^{-2} \times 5^4 = 5^{-2+4} = 5^2 = 25$$

3. Simplify $(-4)^{-4} \times (-4)^2$.

   Solution:

$$(-4)^{-4} \times (-4)^2 = (-4)^{-4+2} = (-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$$

Practice Set

Level 1 – Easy

1. Simplify $3^{-1} \times 3^{-2}$.
2. Simplify 7^0.

Level 2 – Moderate

3. Simplify $(2^{-3} \times 2^5) \div 2^1$.
4. Simplify $(a^3)^{-}2$ where $a \neq 0$.

Level 3 – Challenging

5. Prove that $a^m \times a^{-m} = 1$ for any non-zero a and integer m.
6. Simplify $(-3)^{-4} \times (-3)^{-3}$.

Answer Key

1. $3^{-1} \times 3^{-2} = 3^{-3} = \frac{1}{27}$
2. $7^0 = 1$

3. $(2^{-3} \times 2^5) \div 2^1 = 2^{(-3+5)-1} = 2^1 = 2$
4. $(a^3)^{-2} = a^{3\times(-2)} = a^{-6} = \frac{1}{a^6}$
5. $a^m \times a^{-m} = a^{m-m} = a^0 = 1$
6. $(-3)^{-4} \times (-3)^{-3} = (-3)^{-7} = \frac{1}{(-3)^7}$

Quick Reference

Law	Expression
Product of Powers	$a^m \times a^n = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Product	$a^m \times b^m = (ab)^m$
Zero Exponent	$a^0 = 1$

Glossary

- **Exponent:** The power to which a number is raised.
- **Base:** The number being raised to a power.
- **Negative Exponent:** Indicates reciprocal of the base raised to the positive exponent.
- **Zero Exponent:** Any non-zero number raised to zero is 1.

Standard Form and Comparison of Large and Small Numbers

Standard form (scientific notation) is a way to express very large or very small numbers conveniently using powers of 10.

Concept Explanation

A number is expressed in standard form as:

$$N = a \times 10^n$$

where $1 \leq a < 10$ and n is an integer.

For very large numbers, n is positive; for very small numbers, n is negative.

Formula Derivation

Example: Express 150,000,000,000 in standard form.

Count digits after the first digit: 11 digits.

$$150,000,000,000 = 1.5 \times 10^{11}$$

Example: Express 0.000007 in standard form.

Move decimal point 6 places to the right:

$$0.000007 = 7 \times 10^{-6}$$

Worked Illustrations

- Express 0.0016 in standard form.

$$0.0016 = 1.6 \times 10^{-3}$$

- Express 4,050,000 in standard form.

$$4,050,000 = 4.05 \times 10^{6}$$

Solved Examples

1. Express 0.000035 in standard form.

   Solution:

$$0.000035 = 3.5 \times 10^{-5}$$

2. Express 3.52×10^5 in usual form.

   Solution:

$$3.52 \times 10^{5} = 3.52 \times 100000 = 352000$$

Practice Set

Level 1 – Easy

1. Write 0.000000564 in standard form.
2. Write 21600000 in standard form.

Level 2 – Moderate

3. Write 0.0000021 in standard form.
4. Write 15240000 in standard form.

Level 3 – Challenging

5. Compare the diameter of the Sun (1.4×10^9 m) and the Earth (1.2756×10^7 m).
6. Calculate the total mass of Earth (5.97×10^{24} kg) and Moon (7.35×10^{22} kg).

Answer Key

1. $0.000000564 = 5.64 \times 10^{-7}$
2. $21600000 = 2.16 \times 10^{7}$
3. $0.0000021 = 2.1 \times 10^{-6}$
4. $15240000 = 1.524 \times 10^{7}$
5. Diameter ratio = $\frac{1.4 \times 10^9}{1.2756 \times 10^7} \approx 100$
6. Total mass = $5.97 \times 10^{24} + 7.35 \times 10^{22} = 6.04 \times 10^{24}$ kg

Quick Reference

Number	Standard Form
150,000,000,000	1.5 \times 10^{11}
0.000007	7 \times 10^{-6}

Glossary

- **Standard Form:** Representation of numbers as $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.
- **Exponent:** The power to which 10 is raised in standard form.
- **Scientific Notation:** Another term for standard form.